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Exact ground states and correlation functions of chain and ladder models of interacting hardcore bosons or spinless fermions

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By removing one empty site between two occupied sites, we map the ground states of chains of hardcore bosons and spinless fermions with infinite nearest-neighbor repulsion to ground states of chains of hardcore bosons and spinless fermions without nearest-neighbor repulsion, respectively, and ultimately in terms of the one-dimensional Fermi sea. We then introduce the intervening-particle expansion, where we write correlation functions in such ground states as a systematic sum over conditional expectations, each of which can be ultimately mapped to a one-dimensional Fermi-sea expectation. Various ground-state correlation functions are calculated for the bosonic and fermionic chains with infinite nearest-neighbor repulsion, as well as for a ladder model of spinless fermions with infinite nearest-neighbor repulsion and correlated hopping in three limiting cases. We find that the decays of these correlation functions are governed by surprising power-law exponents.

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I. INTRODUCTION

Exact solutions hold a special place in the theoretical condensed-matter physics of interacting electron systems. Although they can be obtained only for very specific models, these proved to be very useful in understanding the behavior of more general models of interacting electrons or informing us of novel physics that we would otherwise not suspect from approximate treatments. In particular, our present paradigm of two universality classes, Fermi liquids versus Luttinger liquids, for low-dimensional systems of interacting fermions came out of exact solutions showing separation of the charge and spin degrees of freedom.[1–4]

In this paper, we report further surprises coming out of the exact solution of models of hardcore bosons and spinless fermions with infinite nearest-neighbor repulsion.[5] We consider chain models

$$H_{tUV}^{(c,b)} = -t\sum_j [B_j^\dagger B_{j+1} + B_{j+1}^\dagger B_j] + U\sum_j N_j(N_j - \mathbb{1}) + V\sum_j N_j N_{j+1},$$

$$H_{tV}^{(c,f)} = -t\sum_j [C_j^\dagger C_{j+1} + C_{j+1}^\dagger C_j] + V\sum_j N_j N_{j+1} \tag{1}$$

of hardcore bosons ($U\to\infty$) and spinless fermions, as well as a ladder model

$$\begin{aligned} H_{t_\parallel t_\perp t' V}^{(l,f)} = &-t_\parallel \sum_{i=1,2}\sum_j (C_{i,j}^\dagger C_{i,j+1} + C_{i,j+1}^\dagger C_{i,j}) \\ &- t_\perp \sum_j (C_{1,j}^\dagger C_{2,j} + C_{2,j}^\dagger C_{1,j}) \\ &- t'\sum_j (C_{1,j}^\dagger N_{2,j+1} C_{1,j+2} + C_{1,j+2}^\dagger N_{2,j+1} C_{1,j}) \\ &- t'\sum_j (C_{2,j}^\dagger N_{1,j+1} C_{2,j+2} + C_{2,j+2}^\dagger N_{1,j+1} C_{2,j}) \\ &+ V\sum_i\sum_j N_{i,j}N_{i,j+1} + V\sum_i\sum_j N_{i,j}N_{i+1,j} \end{aligned} \tag{2}$$

of spinless fermions. In this ladder model of spinless fermions,[6] the correlated hopping $-t' C_{i,j}^\dagger N_{i'\neq i,j+1} C_{i,j+2}$ is the simplest term we can introduce to blatantly favor the emergence of superconducting order.

Throughout this paper, we will specialize to the limit of infinite onsite repulsion $U\to\infty$ and infinite nearest-neighbor repulsion $V\to\infty$. More precisely, we admit only configurations in which each site can be occupied by at most one particle, with no simultaneous occupation of nearest-neighbor sites. We will show how the *nearest-neighbor excluded chain models* can be mapped to the *nearest-neighbor included chain models*

$$H_{tU}^{(c,b)} = -t\sum_j [b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j] + U\sum_j n_j(n_j - \mathbb{1}),$$

$$H_t^{(c,f)} = -t\sum_j [c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j] \tag{3}$$

and ultimately solve for the ground states of the former in terms of the one-dimensional Fermi sea. We will also show how the ladder model can be solved exactly in three limiting cases by mapping their ground states to those of the chain models given in Eqs. (1) and (3). These analytical results were used to guide a density-matrix analysis of correlations for the ladder model, first using the exactly diagonalized ground states,[7] and later using the density-matrix renormalization group.[8]

For the rest of this paper, we will consistently use uppercase letters B_j and $B_j^\dagger$ (C_j and $C_j^\dagger$) to denote hardcore boson (spinless fermion) annihilation and creation operators on nearest-neighbor excluded chains, and lowercase letters b_j and $b_j^\dagger$ (c_j and $c_j^\dagger$) to denote hardcore boson (spinless fermion) annihilation and creation operators on nearest-neighbor included chains. Similarly, $N_j=B_j^\dagger B_j$ (or $N_j=C_j^\dagger C_j$) and $n_j=b_j^\dagger b_j$ (or $n_j=c_j^\dagger c_j$) are the hardcore boson (spinless fermion) occupation number operator on the nearest-neighbor excluded and nearest-neighbor included chains, respectively. Hereafter, we will also use *excluded* to refer to all quantities associated with the nearest-neighbor excluded chain and *ordinary* to refer to all quantities associated with the nearest-neighbor included chain.

Our paper will be organized as follows: in Sec. II, we will describe an analytical map that establishes a one-to-one correspondence between the Hamiltonian matrices of the excluded and ordinary chains of hardcore bosons and spinless fermions, before developing a systematic expansion that would allow us to calculate ground-state expectations in bosonic and fermionic excluded chains. We then present and analyze in Sec. III correlation functions calculated using the analytical tools developed in Sec. II for excluded chains of hardcore bosons and spinless fermions. Following this, we write down in Sec. IV the exact ground states of the ladder model given in Eq. (2) in three limiting cases and calculate various ground-state correlation functions, before summarizing our results and discuss the interesting physics they imply in Sec. V.

II. MAPPINGS AND TECHNIQUES

In Sec. II A, we establish a one-to-one correspondence between states of the nearest-neighbor excluded and nearest-neighbor included chains. We explain how the Hamiltonian matrices of the two chains, and hence their energy spectra, are identical to one another. In the infinite-chain limit, we then show how we can write the ground state of the excluded chain in terms of the ground state of the ordinary chain, and ultimately be written in terms of the one-dimensional Fermi sea. In Sec. II B, we show how the ground-state expectation between two local operators can be calculated for the excluded chain, by writing it as a systematic sum over conditional expectations, each of which associated with a fixed configuration of intervening particles.

For the sake of definiteness, let us consider open chains of a finite length L and total particle number P. Sites on these chains are indexed by $j=1,\ldots,L$. Since the models in Eqs. (1) and (3) conserve P, the infinite-chain limit is obtained by letting $L\to\infty$ keeping the density of particles $\bar{N}=P/L$ fixed. Ultimately, the results we present in this section will not depend on what boundary conditions we impose on the chain (which is what we would expect in the infinite chain limit).

For convenience, we establish some notations to cover boson and fermion cases together in the same formula. Let us call

$$|\mathbf{J}\rangle \equiv A_{j_1}^\dagger A_{j_2}^\dagger \ldots A_{j_P}^\dagger |0\rangle_L \quad (4)$$

an excluded configuration, where $A=B$ for hardcore bosons, $A=C$ for spinless fermions, and the sites $0<j_1<j_2<\cdots<j_P\leq L$ are such that $j_{p+1}>j_p+1$. We will also employ the labels α and β for distinct P-particle configurations $|\mathbf{J}^\alpha\rangle$ and $|\mathbf{J}^\beta\rangle$; i.e., the P-particle configurations $\{j_1^\alpha,j_2^\alpha,\ldots,j_P^\alpha\}$ and $\{j_1^\beta,j_2^\beta,\ldots,j_P^\beta\}$ differ in at least one site.

Similarly, let us call

$$|\mathbf{j}\rangle \equiv a_{j_1}^\dagger a_{j_2}^\dagger \ldots a_{j_P}^\dagger |0\rangle_L \quad (5)$$

an ordinary configuration, where $a=b$ for hardcore bosons, $a=c$ for spinless fermions, and the sites $0<j_1<j_2<\cdots<j_P\leq L$ are such that $j_{p+1}\geq j_p+1$. The labels α and β will again denote distinct P-particle configurations $|\mathbf{j}^\alpha\rangle$ and $|\mathbf{j}^\beta\rangle$. We will also consistently denote the Hamiltonian of an excluded chain by H_A, where $H_A=H_{tUV}^{(c,b)}$ for hardcore bosons, $H_A=H_{tV}^{(c,f)}$ for spinless fermions, and the Hamiltonian of an ordinary chain by H_a, where $H_a=H_{tU}^{(c,b)}$ for hardcore bosons, $H_a=H_t^{(c,f)}$ for spinless fermions.

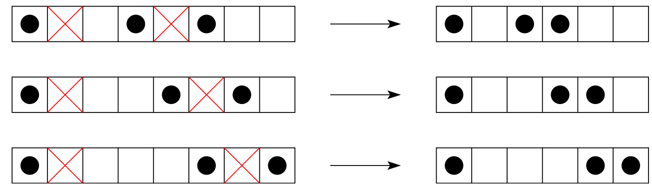

FIG. 1. (Color online) Schematic diagram illustrating how we map from P-particle configurations on an open excluded chain of length L to P-particle configurations on an open ordinary chain of length $L'=L-P+1$, by deleting one empty site to the right of a particle that is not the rightmost particle.

A. Mapping between the excluded and ordinary chains

In this subsection, our goal is to establish the one-to-one correspondence between states of the excluded and ordinary chains, and to show that as matrices, Hamiltonians (1) and (3) are identical. To do this, let us note that an excluded chain with L sites has fewer P-particle states than an ordinary chain of L sites because of the infinite nearest-neighbor repulsion. Therefore, we can form a one-to-one correspondence between excluded and ordinary states only if the length L' of the ordinary chain is shorter than L. There are several ways to systematically map excluded configurations to ordinary configurations: we can (i) delete the site to the right of every particle, if it is not the rightmost particle; or (ii) delete the site to the left of every particle, if it is not the leftmost particle. We can easily check that these maps produce the same ordinary configurations for finite open chains. We expect this to hold true even as we go to the infinite chain limit. For the rest of this paper, we will adopt *right-exclusion map*

$$A_{j_1}^\dagger A_{j_2+1}^\dagger \cdots A_{j_P+P-1}^\dagger |0\rangle \mapsto a_{j_1}^\dagger a_{j_2}^\dagger \cdots a_{j_P}^\dagger |0\rangle \quad (6)$$

that maps a P-particle configuration on an open excluded chain of length L to a P-particle configuration on an open ordinary chain of length L'. The empty site to the right of each occupied site in the open excluded chain is deleted to give a corresponding configuration for an open ordinary chain.[9] As illustrated in Fig. 1, we do not delete any empty site to the right of the Pth particle and thus the effective length of the open ordinary chain is $L'=L-P+1$. This nearest-neighbor exclusion map was first used by Fendley to map a supersymmetric chain of spinless fermions to the *XXZ* chain.[10] It tells us that an excluded chain with density

$$\bar{N}=\frac{P}{L} \tag{7}$$

gets mapped to an ordinary chain with density

$$\bar{n}=\frac{P}{L'}=\frac{P}{L-P+1}=\frac{\bar{N}}{1-\bar{N}+(1/P)}. \tag{8}$$

Thus, in the limit of L, $P\to\infty$,

$$\bar{n}=\frac{\bar{N}}{1-\bar{N}}. \tag{9}$$

This same result was anticipated by Gomez-Santos in his exact solution of generalized spinless fermion chains with short-range interactions that are infinite.[11]

For P-particle excluded configurations $|\mathbf{J}^{\alpha}\rangle$ and $|\mathbf{J}^{\beta}\rangle$, the matrix element $\langle\mathbf{J}^{\alpha}|H_A|\mathbf{J}^{\beta}\rangle$ is nonzero only when $|\mathbf{J}^{\alpha}\rangle$ and $|\mathbf{J}^{\beta}\rangle$ can be obtained from one another by a single particle hopping to the left or the right. When this is so, the ordinary P-particle configurations $|\mathbf{j}^{\alpha}\rangle$ and $|\mathbf{j}^{\beta}\rangle$ they map to under the right-exclusion map are also related to each other by a single particle hop. Thus, we have

$$\langle\mathbf{J}^{\alpha}|H_A|\mathbf{J}^{\beta}\rangle=-t=\langle\mathbf{j}^{\alpha}|H_a|\mathbf{j}^{\beta}\rangle. \tag{10}$$

Since there is a one-to-one correspondence between P-particle configurations on an open excluded chain and P-particle configurations on an open ordinary chain, Eq. (10) tells us that H_A and H_a are identical as matrices in their respective configurational bases. We therefore conclude that the P-particle energy spectra of the two chains coincide and that there is a one-to-one correspondence between the energy eigenstates, $\{|\Psi\rangle\}$ for the excluded chain, and $\{|\Psi'\rangle\}$ for the ordinary chain. That is, if $|\mathbf{J}\rangle\mapsto|\mathbf{j}\rangle$, and $|\Psi\rangle\mapsto|\Psi'\rangle$, then $|\mathbf{J}\rangle$ and $|\mathbf{j}\rangle$ have the same amplitudes in $|\Psi\rangle$ and $|\Psi'\rangle$, respectively. This result has profound implications on the thermodynamics of the two chains, as well as that of the ladder model in the three limiting cases described in Sec. IV A, because their partition functions are the same. However, for the rest of this paper, we limit ourselves to the ground states of the infinite excluded and ordinary chains, as well as those of the infinite ladder.

B. Ground-state expectations of the excluded chain

In this subsection, we explain how the expectation $\langle O\rangle$ of an observable O in the ground state of the excluded chain can be computed, by relating it to the expectation $\langle O'\rangle$ in the ground state of the ordinary chain, for an appropriately chosen observable O' satisfying some basic correspondence requirements that we shall outline. Specifically, we are interested in the correlations $\langle O_1O_2\rangle$ between two local observables O_1 and O_2 separated by a distance r within the excluded-chain ground state. However, the right-exclusion map maps excluded matrix elements $\langle\mathbf{J}^{\alpha}|O_1O_2|\mathbf{J}^{\beta}\rangle$ to ordinary matrix elements $\langle\mathbf{j}^{\alpha}|O_1'O_2'|\mathbf{j}^{\beta}\rangle$ in which O_1' and O_2' are separated by varying separations. To deal with this problem, we develop a method of intervening-particle expansion involving a sum over conditional expectations.

To begin, let us look at the ground states

$$|\Psi_0\rangle=\sum_{\{\mathbf{J}\}}\Psi_0(\mathbf{J})|\mathbf{J}\rangle,$$

$$|\Psi_0'\rangle=\sum_{\{\mathbf{j}\}}\Psi_0'(\mathbf{j})|\mathbf{j}\rangle \tag{11}$$

of the excluded and ordinary chains, respectively. To take advantage of the equality of amplitudes, $\Psi_0(\mathbf{J})=\Psi_0'(\mathbf{j})$ if $|\mathbf{J}\rangle\mapsto|\mathbf{j}\rangle$ under the right-exclusion map, we want $\langle\mathbf{j}^{\alpha}|O'|\mathbf{j}^{\beta}\rangle$ to have a simple relation with $\langle\mathbf{J}^{\alpha}|O|\mathbf{J}^{\beta}\rangle$. While it is possible to pick O' such that $\langle\mathbf{j}^{\alpha}|O'|\mathbf{j}^{\beta}\rangle=\langle\mathbf{J}^{\alpha}|O|\mathbf{J}^{\beta}\rangle$ for all α and β, we find that it is more convenient to pick O' such that

$$\frac{1}{\bar{N}}\langle\mathbf{J}^{\alpha}|O|\mathbf{J}^{\beta}\rangle=\frac{1}{\bar{n}}\langle\mathbf{j}^{\alpha}|O'|\mathbf{j}^{\beta}\rangle. \tag{12}$$

For example, if $O=N_j=C_j^{\dagger}C_j$, we can pick the corresponding observable to be $O'=n_j=c_j^{\dagger}c_j$, in which case we find that

$$\langle\mathbf{J}^{\alpha}|N_j|\mathbf{J}^{\beta}\rangle=\bar{N}\delta_{\alpha\beta},$$

$$\langle\mathbf{j}^{\alpha}|n_j|\mathbf{j}^{\beta}\rangle=\bar{n}\delta_{\alpha\beta}, \tag{13}$$

which satisfies Eq. (12) trivially. We call O and O' a *corresponding pair of observables*, if Eq. (12) is satisfied for all α and β, allowing us to write the very simple relation

$$\frac{1}{\bar{N}}\langle O\rangle=\frac{1}{\bar{n}}\langle O'\rangle \tag{14}$$

between their ground-state expectations.

Since we are mostly interested in correlation functions within the excluded chain ground state, let us look at expectations of the product form $\langle O_jO_{j+r}\rangle$, where O_j acts locally about site j and O_{j+r} acts locally about site $j+r$. Because the number of particles p between sites j and $j+r$ varies from excluded configuration to excluded configuration, these sites get mapped by the right-exclusion map to sites on the ordinary chain with varying separations $r-p$. Therefore, to calculate the excluded chain ground-state expectation $\langle O_jO_{j+r}\rangle$ in terms of ordinary chain ground-state expectations, we first write down an *intervening-particle expansion*

$$\langle O_jO_{j+r}\rangle=\sum_{\mathbf{p}}\langle O_jQ_{\mathbf{p}}O_{j+r}\rangle, \tag{15}$$

where $\langle O_jQ_{\mathbf{p}}O_{j+r}\rangle$ are conditional expectations. Here $\mathbf{p}$ is a vector of occupation numbers within the intervening sites, and $Q_{\mathbf{p}}$ is a string of factors, each of which is either N_{j+s} or $(\mathbb{1}-N_{j+s})$, $1\leq s\leq r-1$. The sum is over all possible ways to have intervening particles between O_j and O_{j+r}. For each excluded term $\langle O_jQ_{\mathbf{p}}O_{j+r}\rangle$ in Eq. (15), we then write down the corresponding ordinary expectation $\langle O_j'Q_{\mathbf{p}'}'O_{j+r-p}'\rangle$, and thereafter sum over all corresponding ordinary expectations,

$$\langle O_j O_{j+r}\rangle = \frac{\bar{N}}{\bar{n}}\sum_{\mathbf{p}'}\langle O'_j Q'_{\mathbf{p}'} O'_{j+r-p}\rangle, \tag{16}$$

making use of Eq. (14). The vector $\mathbf{p}'$ of occupation numbers is obtained from $\mathbf{p}$ using the right-exclusion map and contains the same number p of occupied intervening sites.

To illustrate how the corresponding expectations $\langle O'_j Q'_{\mathbf{p}'} O'_{j+r-p}\rangle$ can be constructed, let us write Eq. (15) out explicitly as

$$\begin{aligned}\langle O_j O_{j+r}\rangle &= \langle O_j(\mathbb{1}-N_{j+1})\cdots(\mathbb{1}-N_{j+r-1})O_{j+r}\rangle \\ &\quad + \langle O_j N_{j+1}\cdots(\mathbb{1}-N_{j+r-1})O_{j+r}\rangle \\ &\quad + \cdots + \langle O_j(\mathbb{1}-N_{j+1})\cdots N_{j+r-1}O_{j+r}\rangle \\ &\quad + \langle O_j N_{j+1}N_{j+2}\cdots(\mathbb{1}-N_{j+r-1})O_{j+r}\rangle \\ &\quad + \cdots + \langle O_j(\mathbb{1}-N_{j+1})\cdots N_{j+r-2}N_{j+r-1}O_{j+r}\rangle \\ &\quad + \cdots + \langle O_j N_{j+1}N_{j+2}\cdots N_{j+r-1}O_{j+r}\rangle, \end{aligned}\tag{17}$$

each of which contains intervening particles at fixed sites. We call terms in the expansion with p intervening N_j's the p*-intervening-particle expectations*. Because of nearest-neighbor exclusion, most of the terms in Eq. (17) vanish.

Next, we map each conditional excluded expectation in Eq. (17) to a corresponding conditional ordinary expectation following the simple rules given below.

(1) Nearest-neighbor exclusion: to ensure that we do not violate nearest-neighbor exclusion, we make the assignment

$$A^\dagger_{j+s}A^\dagger_{j+s+1} = 0 = A_{j+s}A_{j+s+1}. \tag{18}$$

Note that this is intended not as a statement on the operator algebra, but as a mere bookkeeping device for evaluating expectations. The assignment

$$A^\dagger_{j+s}N_{j+s+1} = 0 = N_{j+s}A_{j+s+1} \tag{19}$$

follows from Eq. (18).

(2) Right-exclusion map: the right-exclusion map described in Sec. II A is then implemented by making the substitution

$$A^\dagger_{j+s}(\mathbb{1}-N_{j+s+1}) \mapsto a^\dagger_{j+s}. \tag{20}$$

The assignment

$$N_{j+s}(\mathbb{1}-N_{j+s+1}) = n_{j+s} \tag{21}$$

follows from Eq. (20).

(3) Reindexing: because the right-exclusion map in Eq. (20) merges the occupied site $j+s$ and the empty site $j+s+1$ to its right, operators to the right of site $j+s+1$ must be reindexed. The index $j+s$ on the excluded chain becomes

$$j+s-\sum_{s'=0}^{s-1}N_{j+s'} \tag{22}$$

on the ordinary chain. Thus, two ending operators r sites apart in the p-intervening-particle excluded expectation become $r-p$ sites apart in the corresponding p-intervening-particle ordinary expectation.

III. CORRELATIONS IN THE BOSONIC AND FERMIONIC EXCLUDED CHAINS

In this section, we make use of the tools developed in Sec. II to calculate three simple correlation functions within the ground states of the excluded chains of hardcore bosons and spinless fermions. In general, the intervening-particle expansion for excluded chain ground-state correlations must be evaluated numerically (even when each ordinary chain ground-state expectations in the sum can be expressed in closed form), keeping in mind a excluded chain with density $\bar{N}$ maps to an ordinary chain with density $\bar{n}=\bar{N}/(1-\bar{N})$ [see Eq. (9)].

In Secs. III A–III C, we show numerical results for the two-point functions $\langle B^\dagger_j B_{j+r}\rangle$ and $\langle C^\dagger_j C_{j+r}\rangle$ and the four-point functions $\langle N_j N_{j+r}\rangle$, $\langle B^\dagger_j B^\dagger_{j'} B_{j+r} B_{j'+r}\rangle$, and $\langle C^\dagger_j C^\dagger_{j'} C_{j+r} C_{j'+r}\rangle$, respectively. For the sake of easy reference, we will call the two-point functions $\langle B^\dagger_j B_{j+r}\rangle$ and $\langle C^\dagger_j C_{j+r}\rangle$ Fermi-liquid (FL)-type correlation functions, even though their spatial structures depend on particle statistics. We will also call the four-point functions $\langle N_j N_{j+r}\rangle=\langle B^\dagger_j B_j B^\dagger_{j+r} B_{j+r}\rangle$ and $\langle C^\dagger_j C_j C^\dagger_{j+r} C_{j+r}\rangle$ charge-density wave (CDW)-type correlation functions, and the four-point functions $\langle B^\dagger_j B^\dagger_{j'} B_{j+r} B_{j'+r}\rangle$ and $\langle C^\dagger_j C^\dagger_{j'} C_{j+r} C_{j'+r}\rangle$ superconducting (SC)-type correlation functions. Both CDW- and SC-type correlations are identical for hardcore bosons and spinless fermions on the excluded chain, but the latter has the "superconducting" interpretation only for fermions.

In Sec. III A, we will also explain in detail how nonlinear curve fits of the numerical correlations to reasonable asymptotic forms as a function of r are done. Based on results from the nonlinear curve fits, we show how the Luttinger's theorem manifests itself and how meaningful power-law exponents can be extracted. Similar analyses are done in Secs. III B and III C, as well as in Sec. IV for the three limiting ground states of the ladder model.

A. FL correlations

In the intervening-particle expansions of the two-point functions $\langle B^\dagger_j B_{j+r}\rangle$ and $\langle C^\dagger_j C_{j+r}\rangle$, the nonvanishing terms map to p-intervening-particle expectations of the form $\langle b^\dagger_j \Pi_{j_p} n_{j_p} \Pi_{j_h}(1-n_{j_h}) b_{j+r-p}\rangle$ and $\langle c^\dagger_j \Pi_{j_p} n_{j_p} \Pi_{j_h}(1-n_{j_h}) c_{j+r-p}\rangle$. Both can be evaluated in terms of two-point functions

$$\langle c^\dagger_i c_j\rangle = \frac{\sin \bar{n}\pi|i-j|}{\pi|i-j|} \tag{23}$$

of the one-dimensional Fermi sea, after invoking the Jordan-Wigner transformation (see the Appendix) for the former. At half-filling, both models in Eq. (1) can be mapped to the antiferromagnetic spin-$\frac{1}{2}$ XXZ chain,[12,13] which can be exactly solved by the Bethe ansatz.[14,15] In the $V\to\infty$ Ising limit of the XXZ chain, we know the Bethe ansatz exact solution that the ground state is gapped and hence all correlations should decay exponentially. However, our exact solutions are for chains with $\bar{N}<\frac{1}{2}$, which introduces either a chemical potential μ or an external magnetic field h into the mapped spin model. In the $V\to\infty$ limit, we have $\mu\to\infty$ or $h\to\infty$ as

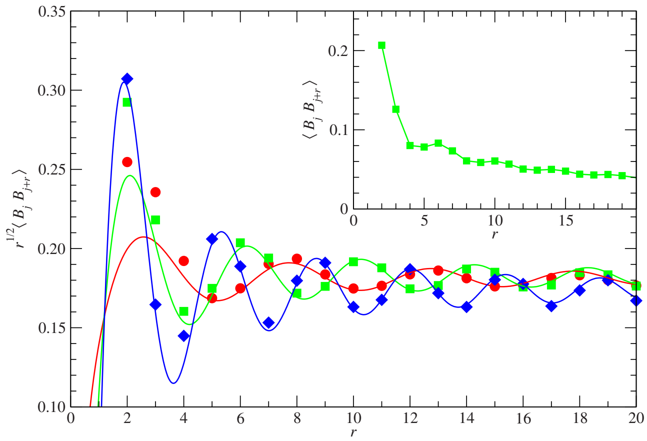

FIG. 2. (Color online) Plot of $\sqrt{r}\langle B_j^\dagger B_{j+r}\rangle$ as a function of r for the particle densities $\bar{N}=0.20$ (red circles), $\bar{N}=0.25$ (green squares), and $\bar{N}=0.30$ (blue diamonds). The colored curves shown are nonlinear curve fits of $\sqrt{r}\langle B_j^\dagger B_{j+r}\rangle$ to the asymptotic form $A_0+A_1 r^{-\alpha_1'}\cos(2k_F r+\phi_1)$. (Inset) Plot of $\langle B_j^\dagger B_{j+r}\rangle$ as a function of r showing that it consists of a simple power-law part and an oscillatory power-law part.

well. Either terms will close the Ising gap, allowing for the emergence of power-law correlations that we find in this subsection.

As shown in the inset of Fig. 2, the FL correlation $\langle B_j^\dagger B_{j+r}\rangle$ was found to consists of a simple power-law part, decaying with a smaller exponent α_0, and an oscillatory power-law part, decaying with a larger exponent α_1. Multiplying $\langle B_j^\dagger B_{j+r}\rangle$ by various simple powers of r, we find that $\sqrt{r}\langle B_j^\dagger B_{j+r}\rangle$ asymptotes to a constant with large r (as shown in the main plot of Fig. 2), which suggests that $\alpha_0=\frac{1}{2}$. This is the correlation exponent predicted by Efetov and Larkin in their study of the ordinary chain of hardcore bosons.[16]

Another important result comes from the unrestricted nonlinear curve fits of $\sqrt{r}\langle B_j^\dagger B_{j+r}\rangle$ to the asymptotic form $A_0+A_1 r^{-\alpha_1'}\cos(kr+\phi_1)$. In Fig. 3 we show a plot of the fitted wave number k as a function of the density $\bar{N}$ of the excluded chain of hardcore bosons. As we can see, the fitted wave numbers fall neatly onto the straight line $k=2k_F=2\pi\bar{N}$, where $k_F=\pi\bar{N}$ is the Fermi wave number (again, anticipated by Gómez-Santos in Ref. 11). The fact that k_F appears naturally in the numerical correlations is expected from Luttinger's theorem, which states that the volume of the reciprocal space bounded by the noninteracting Fermi surface is invariant quantity not affected by interactions and applies in both Fermi and non-Fermi liquids.[2,17–22] From this point onward, we restrict the wave number of the oscillatory part of the correlation functions to k_F, $2k_F$, or $4k_F$ in the nonlinear curve fits.

From the restricted nonlinear curve fits, we find that A_1 is large when α_1' is large and small when α_1' is small. This suggests that neither of these parameters can be accurately determined from our nonlinear curve fits, unless we further constrain what values α_1' can take. We also find that the quality of the nonlinear curve fit is good when $\bar{N}$ is far from $\bar{N}=\frac{1}{4}$ but deteriorates as we approach quarter filling. This

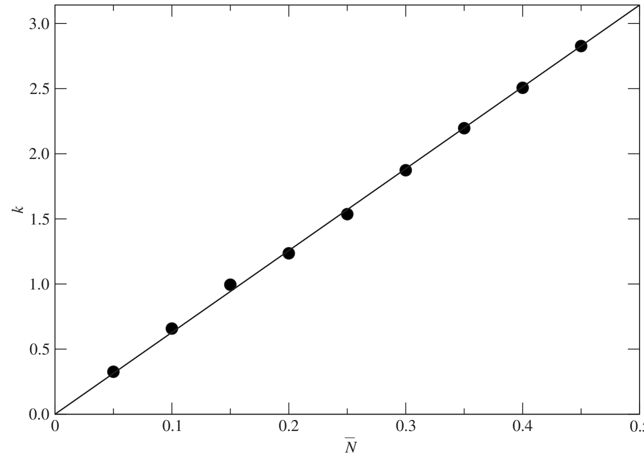

FIG. 3. Plot of the fitted wave number k (solid circles) against the density $\bar{N}$ of the excluded chain of hardcore bosons. The parameter k is obtained from the unrestricted nonlinear curve fit of $\sqrt{r}\langle B_j^\dagger B_{j+r}\rangle$ to the asymptotic form $A_0+A_1 r^{-\alpha_1'}\cos(kr+\phi_1)$. The straight line is $k_F=2\pi\bar{N}$ for the Fermi wave number.

suggests important physics in the FL correlation $\langle B_j^\dagger B_{j+r}\rangle$ near quarter filling, which cannot be adequately accounted for by an asymptotic form $A_0+A_1 r^{-\alpha_1'}\cos(kr+\phi_1)$. This loss of fit also affects the phase shift ϕ_1, presumably to a smaller extent, and the amplitude A_0 of the simple power law, to an even smaller extent. These two parameters are plotted as functions of the density in Fig. 4. In the limit $\bar{N}\to 0$, we have essentially a dilute gas of hardcore (otherwise noninteracting) bosons and thus $\langle B_j^\dagger B_{j+r}\rangle$ should include an overall factor of $\bar{N}$. Thus, we expect $A_0\to 0$ as $\bar{N}\to 0$. In the limit $\bar{N}\to\frac{1}{2}$, the excluded chain of hardcore bosons becomes increasingly jammed and the relevant degrees of freedom are holes. For a dilute chain of holes, we expect $\langle B_j^\dagger B_{j+r}\rangle$ to be proportional to the hole density and thus $A_0\to 0$ as $\bar{N}\to\frac{1}{2}$. From our numerical results alone, it is hard to tell whether A_0

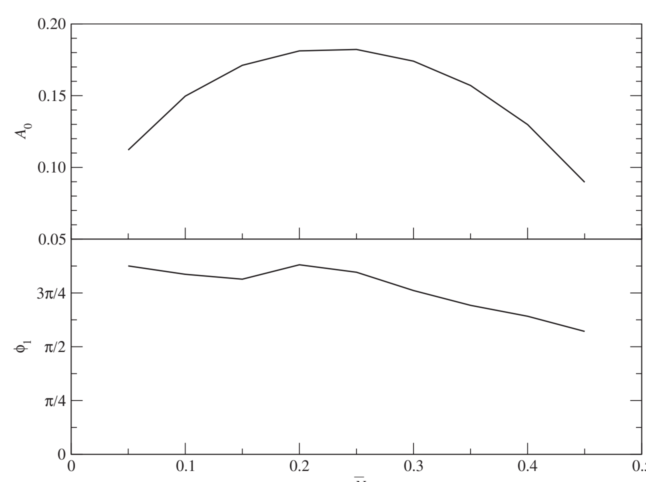

FIG. 4. Plot of the fitted amplitude A_0 (top) of the simple power-law part and the fitted phase shift ϕ_1 (bottom) of the oscillatory power-law part as functions of the density $\bar{N}$ of the excluded chain of hardcore bosons.

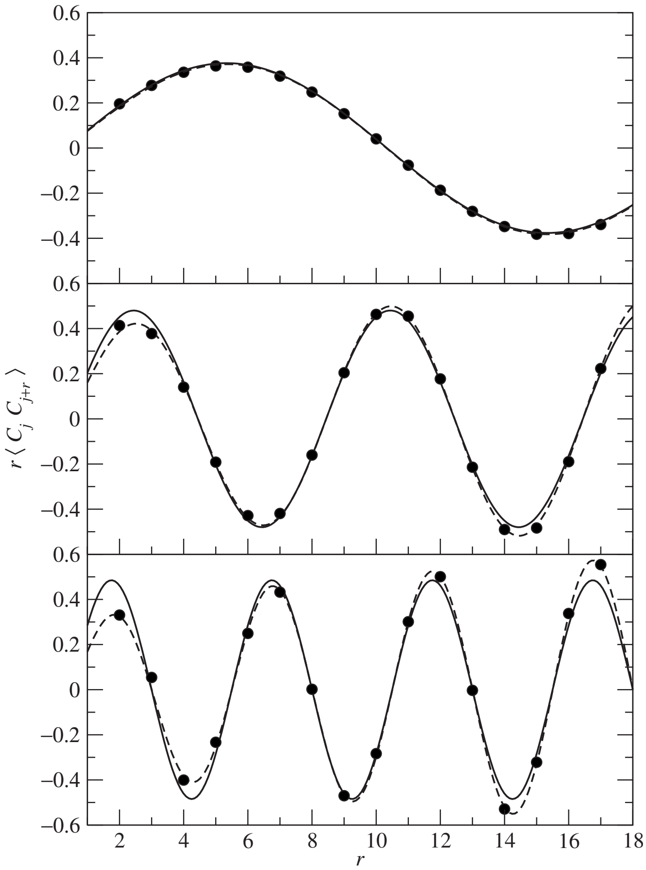

FIG. 5. Plots of $r\langle C_j^\dagger C_{j+r}\rangle$ as functions of r (solid circles) for particle densities $\bar{N}=0.10$ (top), $\bar{N}=0.25$ (middle), and $\bar{N}=0.40$ (bottom). Nonlinear curve fits of $r\langle C_j^\dagger C_{j+r}\rangle$ to the restricted asymptotic form $A_1\cos(\pi\bar{N}r+\phi_1)$ (solid curves) show systematic deviations, which can be accounted for by a unrestricted asymptotic form $A_1 r^{(1-\alpha_1)}\cos(\pi\bar{N}r+\phi_1)$.

reaches a maximum at $\bar{N}=\frac{1}{5}$ (corresponding to a quarter-filled, $\bar{n}=\frac{1}{4}$, ordinary chain of hardcore bosons) or $\bar{N}=\frac{1}{4}$ (quarter-filled excluded chain of hardcore bosons). It is also hard to say anything definite about the phase shift ϕ_1, which might in fact be constant.

In contrast, the FL correlation $\langle C_j^\dagger C_{j+r}\rangle$ contains no simple power-law part. A preliminary unrestricted nonlinear curve fit of this correlation to the asymptotic form $A_1 r^{-\alpha_1}\cos(\pi\bar{N}r+\phi_1)$ suggests that $\alpha_1\approx 1$ for all densities, i.e., may be universal $\alpha_1=1$ just as for noninteracting spinless fermions. However, a more careful restricted nonlinear curve fit $r\langle C_j^\dagger C_{j+r}\rangle = A_1\cos(\pi\bar{N}r+\phi_1)$ show systematic deviations, as shown in Fig. 5, and therefore we perform an unrestricted fit to $r\langle C_j^\dagger C_{j+r}\rangle = A_1 r^{1-\alpha_1}\cos(\pi\bar{N}r+\phi_1)$. The fitted parameters A_1, $1-\alpha_1$, and ϕ_1 are shown in Fig. 6.

At very low densities $\bar{N}\to 0$, our dilute chain of nearest-neighbor excluding spinless fermions will behave like noninteracting fermions, so we expect

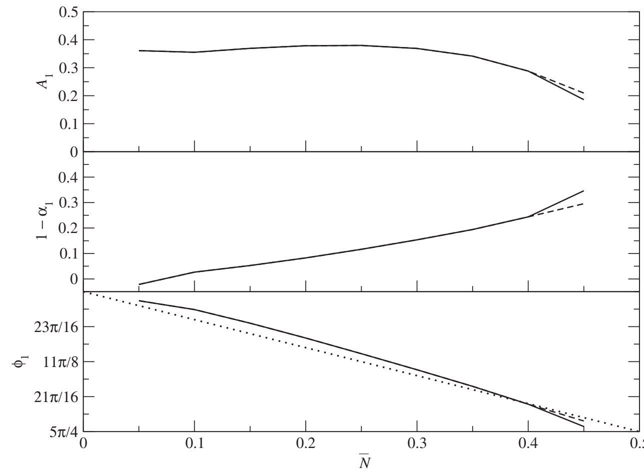

FIG. 6. Plot of the fitted amplitude A_1 (top), the fitted exponent α_1 (middle), and the fitted phase shift ϕ_1 of the leading oscillatory power-law decay in the FL correlation $\langle C_j^\dagger C_{j+r}\rangle$ as functions of the density $\bar{N}$ of the excluded chain of spinless fermions. In these plots, the solid curves are for fits to $A_1 r^{1-\alpha_1}\cos(\pi\bar{N}r+\phi_1)$, whereas the dashed curves are for fits to $A_1 r^{1-\alpha_1}\cos(\pi\bar{N}r+\phi_1) + A_2 r^{1-\alpha_2}\cos[\pi(1-\bar{N})r+\phi_2]$. The dotted line in the bottom plot is a straight line from $5\pi/4$ at $\bar{N}=0$ to $3\pi/2$ at $\bar{N}=\frac{1}{2}$ to guide the eye.

$$\langle C_j^\dagger C_{j+r}\rangle \approx \frac{\sin \pi\bar{N}r}{\pi r}. \tag{24}$$

From our curve fits, we see that in this limit, $A_1\to 1/\pi = 0.318\,31\ldots$, $1-\alpha_1\to 0$, and $\phi_1\to 3\pi/2$, and thus the FL correlation does indeed go to the low density limit in Eq. (24).

Also, in the half-filling limit $\bar{N}\to\frac{1}{2}$, the chain becomes more and more congested, making it increasingly difficult to annihilate a spinless fermion at site $j+r$, find an empty site j to create a spinless fermion, without running afoul of the nearest-neighbor exclusion constraint. This tells us that A_1 must vanish as $\bar{N}\to\frac{1}{2}$, which is hinted at in Fig. 6. However, the vanishing amplitude is only half of the story in this limit, the other half being the rate at which the correlation decays with increasing separation. In fact, very close to $\bar{N}=\frac{1}{2}$, we expect the ground-state physics of the chain of rung fermions with infinite nearest-neighbor repulsion to be describable in terms of a low density of holes. Naively, we would expect from such a low-density-of-holes argument that $\langle C_j^\dagger C_{j+r}\rangle$ decay as r^{-1}. Instead, the nonlinear curve fits at $\bar{N}_1\lesssim\frac{1}{2}$ tell us that $\alpha_1<1$.

Thinking about this nearly-half-filled limit more carefully, we realized that what we called "holes" are really domain walls separating a region in which the spinless fermions sit on odd sites, from a region in which the spinless fermions sit on even sites. The FL correlation $\langle C_j^\dagger C_{j+r}\rangle$, which can be written as a hole-hole correlation function, then depends on how many holes there are between j and $j+r$. The idea is that, in order to annihilate a hole (create a spinless fermion) at site $j+r$ and create a hole (annihilate a spinless fermion) at site j, we must first find a configuration with a hole at $j+r$.

Such a configuration will have spinless fermions at sites $j+r-2$, $j+r-4,\ldots$, until we encounter another hole at $j+r-2s$, and then the sequence of spinless fermions will thereafter be at sites $j+r-2s-1$, $j+r-2s-3,\ldots$. If r is even, $\langle C_j^\dagger C_{j+r}\rangle$ receives nonzero contributions only from those configurations with an even number of intervening holes, whereas if r is odd, $\langle C_j^\dagger C_{j+r}\rangle$ receives nonzero contributions only from those configurations with an odd number of intervening holes. This is very similar in flavor to the intervening-particle expansion of the two-point function $\langle b_j^\dagger b_{j+r}\rangle$ of a chain of ordinary hardcore bosons, except that $\langle b_j^\dagger b_{j+r}\rangle$ receives positive contributions from configurations with an even number of intervening particles and negative contributions from an odd number of intervening particles. Therefore, in the limit $\bar{N}\to\frac{1}{2}$, we find that the FL correlation $\langle C_j^\dagger C_{j+r}\rangle$ maps to a string correlation of holes. Bosonization calculations then show that this string correlation of holes decays as a power law with correlation exponent $\alpha_1=\frac{1}{4}$.[23]

B. CDW correlations

Important physics can also be learnt from the nonlinear curve fitting of the CDW correlations $\langle N_jN_{j+r}\rangle \equiv\langle B_j^\dagger B_j B_{j+r}^\dagger B_{j+r}\rangle=\langle C_j^\dagger C_j C_{j+r}^\dagger C_j\rangle$. First we tried to fit the subtracted CDW correlation to the asymptotic form $\langle N_jN_{j+r}\rangle-\langle N_j\rangle\langle N_{j+r}\rangle=B_1 r^{-\beta_1}\cos(2\pi\bar{N}r+\theta_1)$, but found the quality of fit deteriorates as $\bar{N}\to 0$, as shown in Fig. 7. We understand this as follows: for $\bar{N}\to 0$, the dimensionless quantity $\xi=\bar{N}r$ is small and the poor fit indicates that $\langle N_jN_{j+r}\rangle$ contains contributions from a term that decays more rapidly than $B_1 r^{-\beta_1}\cos(2\pi\bar{N}r+\theta_1)$. If we assume that this faster decaying term is a simple power law of the form $B_2 r^{-\beta_2}$ and fit the CDW correlation to $\langle N_jN_{j+r}\rangle=B_1 r^{-\beta_1}\cos(2\pi\bar{N}r+\theta_1)+B_2 r^{-\beta_2}$, we find that the quality of the nonlinear curve fit is improved, after dropping data points $r=2,3$ from the fit, as shown in Fig. 7. We can include the simple power-law decay term in the nonlinear curve fit throughout the entire range of $\bar{N}$ but the parameters B_2 and β_2 cannot be reliably determined beyond $\bar{N}=\frac{1}{4}$. Therefore, the parameters B_1, β_1, and θ_1 are the only parameters that can be reliably determined across the whole range of densities. From Fig. 8, we find that θ_1 changes very little over the whole range of densities and remains close to $\pi/16$. On the other hand, the leading correlation exponent β_1 appears to be density dependent and is very close to being

$$\beta_1=\tfrac{1}{2}+\tfrac{5}{2}\left(\tfrac{1}{2}-\bar{N}\right). \tag{25}$$

Just as for the FL correlation, we need to again think hard about the hole physics of the ground-state CDW correlation very close to half-filling. In this limit, the CDW correlation $\langle N_jN_{j+r}\rangle$ can be written in terms of the hole-density-hole-density correlation $\langle P_jP_{j+r}\rangle$, where $P_j=\mathbb{1}-N_j$ is the hole occupation number operator on site j. Using an argument similar to the one used for the FL correlation $\langle C_j^\dagger C_{j+r}\rangle$ near half-filling, we realized that if r is even, $\langle P_jP_{j+r}\rangle$ receives nonzero contribution from configurations with an even number of intervening holes, whereas if r is odd, $\langle P_jP_{j+r}\rangle$ receives nonzero contribution from configurations with an odd number of intervening holes. This means that our hole-density-hole-density correlation $\langle P_jP_{j+r}\rangle$ must be mapped to a string correlation of a chain of noninteracting spinless fermions. Bosonization calculations tell us that this string correlation decays as a power law with correlation exponent $\alpha=\frac{1}{2}$ consistent with the conjectured behavior, Eq. (25), at the special value $\bar{N}=\frac{1}{2}$.[23]

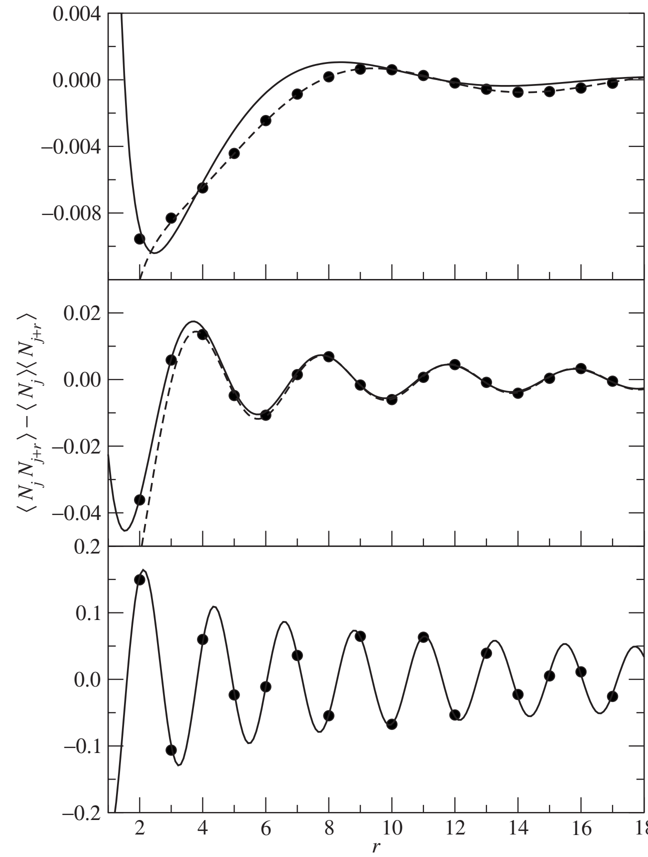

FIG. 7. Nonlinear curve fits of the subtracted CDW correlation $\langle N_jN_{j+r}\rangle$ (solid circles) in the bosonic or fermionic ground states of the excluded chain at densities $\bar{N}=0.10$ (top), $\bar{N}=0.25$ (middle), and $\bar{N}=0.45$ (bottom). Above quarter filling (middle and bottoms plots), the subtracted CDW correlations can be fitted very well to the simple asymptotic form $B_1 r^{-\beta_1}\cos(2\pi\bar{N}r+\theta_1)$ (solid curves), whereas at low densities (top plot), the subtracted CDW correlations deviate significantly from this simple asymptotic form. The nonlinear curve fit improves only after we add a simple power-law correction term $B_2 r^{-\beta_2}$ giving the dashed curves.

C. SC correlations

In contrast to the FL and CDW correlations, the SC correlation $\langle A_{j-2}^\dagger A_j^\dagger A_{j+r}A_{j+r+2}\rangle$ on the excluded chain has a rather more complex structure. The SC correlation is always negative and oscillations are highly suppressed suggesting that it is the sum of a simple power law and an oscillatory power law. To improve the reliability of the nonlinear curve fits, we prescale the SC correlation by multiplying it by $r^{7/4}$. This strange exponent is chosen because it is closest to the rate at

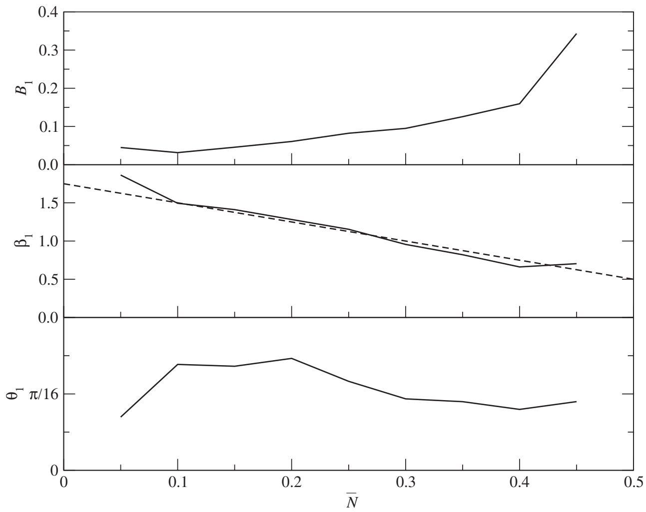

FIG. 8. Plot of the fitted amplitude B_1 (top), the fitted exponent β_1 (middle), and the fitted phase shift θ_1 (bottom) of the leading oscillatory power-law decay of the subtracted CDW correlation $\langle N_j N_{j+r}\rangle - \langle N_j\rangle\langle N_{j+r}\rangle$ as functions of the density $\bar{N}$ of the excluded chain of hardcore bosons or spinless fermions.

which the simple power law decays for various densities. After dropping data points for $r=2,3,4$, good fits to the asymptotic form $r^{7/4}\langle A_{j-2}^{\dagger}A_j^{\dagger}A_{j+r}A_{j+r+2}\rangle = C_0 r^{7/4-\gamma_0} + C_1 r^{7/4-\gamma_1}\cos(2\pi\bar{N}r+\chi_1)$ were obtained. The nonlinear curve fits were improved marginally by adding a correction term of the form $C_2 r^{7/4-\gamma_2}\cos(4\pi\bar{N}r+\chi_2)$ (see Fig. 9). The fitted parameters are shown in Fig. 10 as functions of the excluded chain density $\bar{N}$.

Unlike for the FL and CDW correlations, there are no analytical bosonization calculations to help suggest values for the SC correlation exponents, so we used an *ad hoc* process where we imposed trial values of the exponents and let the nonlinear curve fitting program find the appropriate amplitudes and phase shifts. We found visually that the best fit of the numerical SC correlations appears to the mixed asymptotic form

$$r^{7/4}\langle A_{j-2}^{\dagger}A_j^{\dagger}A_{j+r}A_{j+r+2}\rangle = C_0' r^{-1/8} + C_1' r^{-1/4}\cos(2\pi\bar{N}r + \chi_1') + C_2' r^{-3/2}\cos(2\pi\bar{N}r + \chi_2') + C_3' r^{-7/2}. \tag{26}$$

IV. LADDER MODEL

In this section we show how the analytical machinery developed in Sec. II can be adapted to calculate ground-state correlations in the ladder model of interacting spinless fermions given in Eq. (2), in three limiting cases where the ground states can be deduced from simple energetic arguments. An overview is given in Sec. IV A, before we move on to detailed analyses and discussions of the three limiting cases in Secs. IV B–IV D. As with the chain models, we assume that the ladder is finite, with $j=1,\ldots,L$, and subject each of its legs $i=1,2$ to open boundary conditions. Exact

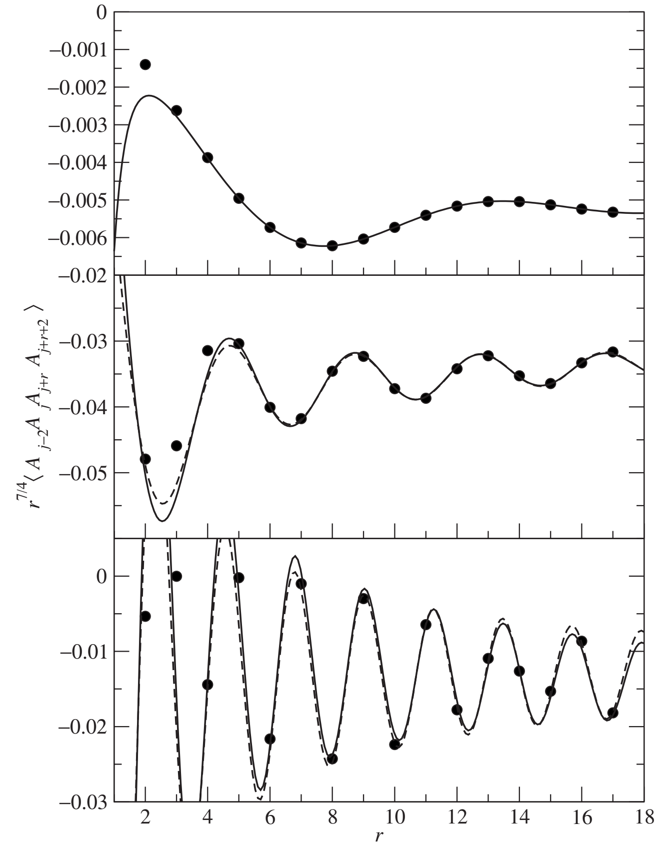

FIG. 9. Nonlinear curve fits of the SC correlation $r^{7/4}\langle A_{j-2}^{\dagger}A_j^{\dagger}A_{j+r}A_{j+r+2}\rangle$ (solid circles) in the bosonic or fermionic ground states of the excluded chain at densities $\bar{N}=0.10$ (top), $\bar{N}=0.25$ (middle), and $\bar{N}=0.45$ (bottom). After dropping data points from $r=2,3,4$, the SC correlations can be fitted well to the simple asymptotic form $C_0 r^{7/4-\gamma_0}+C_1 r^{7/4-\gamma_1}\cos(2\pi\bar{N}r+\chi_1)$ (solid curves). The nonlinear curve fits are marginally improved by adding a correction term of the form $C_2 r^{7/4-\gamma_2}\cos(4\pi\bar{N}r+\chi_2)$ (dashed curves).

solution for the infinite ladder is then obtained by taking $L\to\infty$ keeping the particle density $\bar{N}_2$ fixed. Just as for the chain models, we expect in this limit that the ladder exact solutions would not depend on which boundary conditions we used.

A. Three limiting cases: An overview

For the ladder model described by Eq. (2), with $V\to\infty$, the ground state is determined by the two independent model parameters, $t_{\perp}/t_{\parallel}$ and $t'/t_{\parallel}$, and the density $\bar{N}_2$. For fixed $\bar{N}_2$, the two-dimensional region in the ground-state phase diagram is bounded by three limiting cases as follows.

(i) The *paired limit* $t'\gg t_{\parallel}, t_{\perp}$, which we will discuss in detail in Sec. IV B: in this limit, we find SC correlations dominating at large distances (though, as for hardcore bosons, CDW correlations inevitably dominate at short distances). Based on our numerical studies in Sec. IV B, the leading SC correlation exponent appears to be universal, with a value of $\gamma=\frac{1}{2}$, while the leading CDW correlation exponent β is nonuniversal. In this limit, FL correlations are

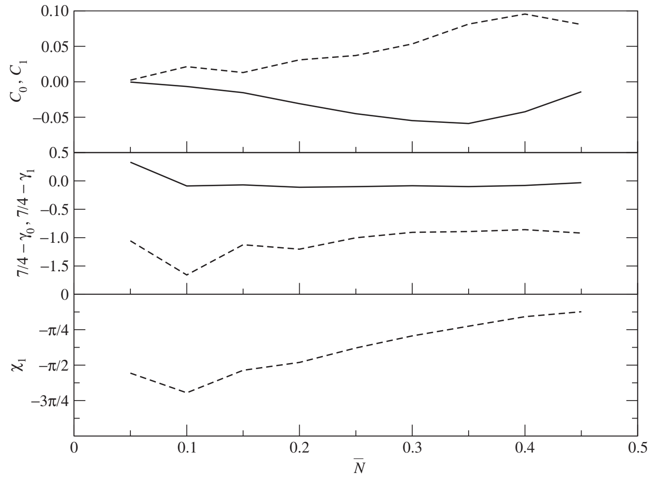

FIG. 10. Plot of the fitted amplitudes C_0 and C_1 (top), the fitted exponents γ_0 and γ_1 (middle), and the fitted phase shift χ_1 of the leading simple power-law decay and the subleading oscillatory power-law decay in the SC correlation $\langle A^\dagger_{j-2}A^\dagger_j A_{j+r}A_{j+r+2}\rangle$, as functions of the density $\bar{N}$ of the excluded chain of hardcore bosons or spinless fermions.

found to decay exponentially. A staggered form of long-range CDW order also appears.

(ii) The *two-leg limit* $t_\perp \ll t_\parallel$, $t'=0$, which we will discuss in detail in Sec. IV C: in this limit, the two legs of the ladder are coupled only by infinite nearest-neighbor repulsion. The dominant correlations at large distances are those of a power-law CDW, for which we find numerically to have what appears to be an universal correlation exponent of $\beta=\frac{1}{2}$. In this limit, the leading SC correlation exponent was predicted analytically to be $\gamma=2$, while FL correlations are found to decay exponentially.

(iii) The *rung-fermion limit* $t_\perp \gg t_\parallel$, $t'=0$, which we will discuss in detail in Sec. IV D: in this limit, the particles are effectively localized onto the rungs of the ladder. When the ladder is quarter filled, a true long-range CDW emerges in the twofold degenerate ground state. Below quarter filling, we find numerically that the CDW power-law correlation dominate at large distances with a leading nonuniversal correlation exponent $\beta=\frac{1}{2}+\frac{5}{2}(\frac{1}{2}-\bar{N}_1)$. The leading FL correlation exponent was also found numerically to be nonuniversal, with values going from $\alpha=1$ to $\alpha=\frac{1}{4}$. The SC correlation exponent, on the other hand, was found numerically to be universal, with value $\gamma=\frac{7}{4}$.

To zeroth order (i.e., without plunging into first-order perturbation theory calculations), the ground-state phase diagram can be obtained by interpolating between these three limiting cases. There will be three lines of quantum phase transitions or crossovers, which at quarter filling, separate the long-range CDW (LR-CDW), power-law CDW (PL-CDW), and SC phases. At quarter filling, we expect these three lines of critical points or crossovers to meet at a point on the phase diagram. If we have three lines of true critical points, this point would be a quantum tricritical point. We therefore end up with a ground-state phase diagram which looks like that shown in Fig. 11.

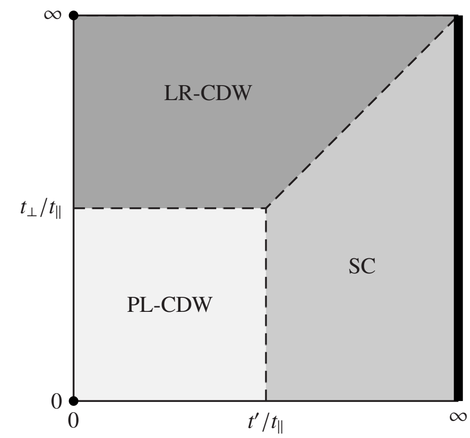

FIG. 11. The zeroth-order ground-state phase diagram of the ladder model given by Eq. (2). The three limiting cases we can solve exactly are shown as the two dots [cases (ii), power-law CDW (PL-CDW) and (iii), long-range CDW (LR-CDW)] and the thick solid line [case (i), SC].

B. Paired limit

In this subsection, we solve for the ground-state wave function and calculate various ground-state correlations in the paired limit $t' \gg t_\parallel$, $t_\perp$. In this limit, the Hamiltonian in Eq. (2) simplifies to

$$\begin{aligned} H_{t'V} = &-t'\sum_i\sum_j (c^\dagger_{i,j}n_{i+1,j+1}c_{i,j+2} + c^\dagger_{i,j+2}n_{i+1,j+1}c_{i,j}) \\ &-t'\sum_i\sum_j (c^\dagger_{i+1,j}n_{i,j+1}c_{i+1,j+2} + c^\dagger_{i+1,j+2}n_{i,j+1}c_{i+1,j}) \\ &+V\sum_i\sum_j n_{i,j}n_{i,j+1} + V\sum_i\sum_j n_{i,j}n_{i+1,j}. \end{aligned} \quad (27)$$

In Sec. IV B 1, we explain how pairs of spinless fermions are bound by correlated hops in this limit and the degrees of freedom in the system become mobile bound pairs with infinite nearest-neighbor repulsion. These bound pairs come in two flavors determined by the specific arrangement of the two bound-pair particles around a plaquette. These flavors are conserved by correlated hops if the length of the ladder is even, and hence the ladder ground state is twofold degenerate. We then describe how these two degenerate ladder ground states can be mapped to an excluded chain of hardcore bosons, then to an ordinary chain of hardcore bosons, and finally to a chain of noninteracting spinless fermions.

In Sec. IV B 2, we calculate the SC and CDW correlations using the intervening-particle expansion described in Sec. II B. We then use a restricted-probability argument in Sec. IV B 3 to show that FL correlations decay exponentially with distance, governed by a density-dependent correlation length, in this paired limit. We find, as expected from making the absolute correlated hopping amplitude t' large, that SC correlations dominate at large distances.

1. Bound pairs and ground states

In the paired limit $t' \gg t_\parallel$, $t_\perp$, we solve for the ground state of the simplified Hamiltonian given by Eq. (27), which ad-

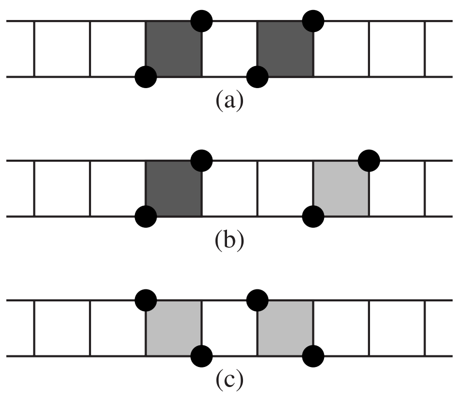

FIG. 12. The closest approach two bound pairs can make to each other, if (a) they both have even flavors; (b) they have opposite flavors; and (c) they both have odd flavors.

mits only correlated hops. Because of this, isolated spinless fermions cannot hop at all; by contrast, a pair occupying diagonal corners on a plaquette can perform correlated hops. Therefore, for an even number of spinless fermions, ground-state configurations consist of well-defined bound pairs, which are effectively bosons. We say that a bound pair at $(1,j)$ and $(2,j+1)$ has even (respectively, odd) flavor if its two sites are even (respectively, odd) sites. In this limit of $t'/t_{\parallel}$, $t'/t_{\perp}\to\infty$, a particle on rung j can only hop to rung $j\pm 2$. This moves the bound pair's center of mass by one lattice constant without changing its flavor. The degrees of freedom in this limiting case thus become bound pairs with definite flavors hopping along a one-dimensional chain.[24] We write these hardcore boson operators in terms of the spinless fermion operators as

$$B_{j,+}^{\dagger}=\begin{cases}c_{1,j}^{\dagger}c_{2,j+1}^{\dagger}, & j\ \text{odd}\\ c_{1,j+1}^{\dagger}c_{2,j}^{\dagger}, & j\ \text{even},\end{cases} \tag{28}$$

and

$$B_{j,-}^{\dagger}=\begin{cases}c_{1,j+1}^{\dagger}c_{2,j}^{\dagger}, & j\ \text{odd}\\ c_{1,j}^{\dagger}c_{2,j+1}^{\dagger}, & j\ \text{even},\end{cases} \tag{29}$$

where we order first with respect to the leg index and then with respect to the rung index of the ladder.

Since bound pairs cannot move past each other along the chain, the P-bound-pair Hilbert space breaks up into many independent sectors, each with a fixed sequence of flavors. The P-bound-pair problem in one sector is therefore an independent problem from that of another P-bound-pair sector. The minimum energy in each sector can be very crudely determined by treating the P-bound-pair problem as a particle-in-a-box problem, where each bound pair is free to hop within a "box" demarcated by its flanking bound pairs.

As shown in Fig. 12, two bound pairs with the same flavor can get within a separation $r=2$ of each other, whereas two bound pairs with different flavors can only achieve a closest approach with separation $r=3$. Therefore, for a fixed separation between flanking bound pairs, the kinetic energy of the "boxed" bound pair is lowest when all three bound pairs have the same flavor. Repeating this argument for all bound pairs, we realized, therefore, that the twofold degenerate ground state lies within the all-even and all-odd sectors. In these sectors, bound pairs cannot occupy nearest-neighbor plaquettes; i.e., we are dealing with an excluded chain of hardcore bosons.[25]

The twofold degeneracy between all-even and all-odd sectors represents a symmetry breaking with long-range order of a staggered CDW type (in terms of fermion densities). It may be viewed as breaking the invariance under reflection about the ladder axis of the original Hamiltonian as given in Eq. (2).[26] Thus, the quantum-mechanical problem of a ladder with density $\bar{N}_2$ is mapped to the quantum-mechanical problem of an excluded chain with density $\bar{N}=\bar{N}_2$.

2. SC and CDW correlations

Three simple correlation functions, the FL, CDW, and SC correlations, were computed for the excluded chain of hardcore bosons in Sec. III. On the ladder model in the paired limit, these correlations must be interpreted differently. In mapping the ladder model to the excluded chain, we replace a pair of spinless fermion by a hardcore boson operator, i.e., $c_{1,j}^{\dagger}c_{2,j+1}^{\dagger}\to B_j^{\dagger}$. Thus, the FL correlation $\langle B_j^{\dagger}B_{j+r}\rangle$ of hardcore bosons actually corresponds to a SC correlation of the fermion model on the ladder. Depending on which of the two degenerate ladder ground states we are looking at, the SC operators are

$$\Delta_{j,g}^{\dagger}=\frac{1}{\sqrt{2}}(c_{1,j}^{\dagger}c_{2,j+1}^{\dagger}+c_{1,j+1}^{\dagger}c_{2,j}^{\dagger}),$$

$$\Delta_{j,u}^{\dagger}=\frac{1}{\sqrt{2}}(-1)^{j}(c_{1,j}^{\dagger}c_{2,j+1}^{\dagger}-c_{1,j+1}^{\dagger}c_{2,j}^{\dagger}), \tag{30}$$

such that

$$\langle\Delta_{j,g}^{\dagger}\Delta_{j+r,g}\rangle_u=\langle\Delta_{j,u}^{\dagger}\Delta_{j+r,u}\rangle_g=0,$$

$$\langle\Delta_{j,g}^{\dagger}\Delta_{j+r,u}\rangle_g=\langle\Delta_{j,g}^{\dagger}\Delta_{j+r,u}\rangle_u=0. \tag{31}$$

Because of Eq. (31), we shall drop the indices g and u from here on. From Sec. III, we know that $\langle\Delta_j^{\dagger}\Delta_{j+r}\rangle$ decays with separation r asymptotically as the sum of a simple (leading) power law and an $2k_F$-oscillatory (subleading) power law. The leading correlation exponent has been determined to be $\gamma_0=\frac{1}{2}$, while the subleading correlation exponent γ_1 cannot be reliably determined.

The simplest CDW correlations are

$$\langle c_{1,j}^{\dagger}c_{1,j}c_{1,j+r}^{\dagger}c_{1,j+r}\rangle,\quad \langle c_{1,j}^{\dagger}c_{1,j}c_{2,j+r}^{\dagger}c_{2,j+r}\rangle,$$

$$\langle c_{2,j}^{\dagger}c_{2,j}c_{1,j+r}^{\dagger}c_{1,j+r}\rangle,\quad \langle c_{2,j}^{\dagger}c_{2,j}c_{2,j+r}^{\dagger}c_{2,j+r}\rangle, \tag{32}$$

which we call the CDW-σ correlations. These are not easy to calculate because they cannot be written simply in terms of the expectations of hardcore boson operators. In contrast, the CDW-π correlations[27]

$$\langle B_j^{\dagger}B_jB_{j+r}^{\dagger}B_{j+r}\rangle=\langle N_jN_{j+r}\rangle \tag{33}$$

can be evaluated with the help of the intervening-particle expansion in Eq. (17). This was done in Sec. III, where we

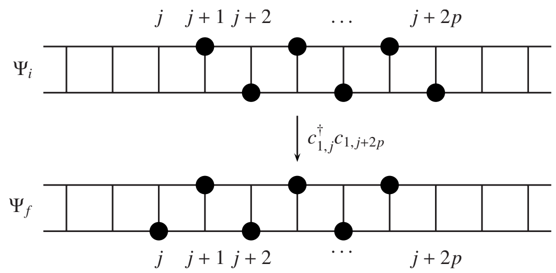

FIG. 13. Compact p-bound-pair cluster configurations making nonzero contribution toward the expectation of the FL operator product $c^{\dagger}_{1,j}c_{1,j+2p}$, which annihilates a spinless fermion on the right end of the compact p-bound-pair cluster and creates a spinless fermion on the left end of the compact p-bound-pair cluster. Ψ_i and Ψ_f are the ground-state amplitudes of the initial and final configurations, respectively.

found the subtracted CDW-π correlation $\langle N_j N_{j+r}\rangle - \langle N_j\rangle\langle N_{j+r}\rangle$ decaying asymptotically with separation r as a simple power law $B_1 r^{-\beta_1}\cos(2k_F r+\theta_1)$, with a nonuniversal leading correlation exponent $\beta_1=\frac{1}{2}+\frac{5}{2}(\frac{1}{2}-\bar{N}_2)$, and a universal phase shift of $\theta_1=\pi/16$.

3. FL correlation: Explanation of exponential decay

Unlike the SC and CDW-π correlations, the FL correlations cannot be calculated easily in this paired limit because the operators involved cannot be written in terms of hardcore boson operators. Nevertheless, we can still calculate it by making use of the fact that this correlation is very close to being the probability of finding a restricted class of configurations in the ground state. We then make use of the scaling form reported in Ref. 28 to calculate the probability analytically. This idea is exploited again in Sec. IV C 2

In this paired limit, the ground state consists exclusively of a superposition of bound-pair configurations. Therefore, if we annihilate a spinless fermion on leg i, we must create another on the same leg elsewhere, and thus the only nonzero FL correlations are of the form $\langle c^{\dagger}_{i,j}c_{i,j+r}\rangle$. In fact, to start with a paired configuration and end up with another paired configuration, after annihilating a spinless fermion at $j+r$ and creating a spinless fermion at j, the initial and final configurations must contain a compact cluster of pairs between rung j and rung $j+r$, as shown in Fig. 13.

Based on this compact cluster argument, we know that $\langle c^{\dagger}_{i,j}c_{i,j+r}\rangle=0$ when r is odd. When $r=2p$ is even,

$$\langle c^{\dagger}_{i,j}c_{i,j+2p}\rangle = \sum_{(i,f)} \Psi_f^{*}\Psi_i \tag{34}$$

receives contributions from all pairs of configurations with a compact p-bound-pair cluster between the rungs j and $j+2p$. Clearly, these products of amplitudes will depend on where the other bound pairs are on the ladder. However, if the ladder is not too close to half-filling, we expect $\Psi_f\approx\Psi_i$, so that on an infinite ladder, $\langle c^{\dagger}_{i,j}c_{i,j+2p}\rangle$ is very nearly the probability of finding a compact p-bound-pair cluster,[29]

$$\langle N_j N_{j+2}\cdots N_{j+2p}\rangle = \frac{\bar{N}_2}{\bar{n}}\langle n_j n_{j+1}\cdots n_{j+p}\rangle = \frac{\bar{N}_2}{\bar{n}}\det G_C(p), \tag{35}$$

after using relation (14) between excluded and ordinary expectations, where $\bar{n}=\bar{N}_2/(1-\bar{N}_2)$ is the density of the ordinary chain. Here $\det G_C(p)$ is the determinant of the noninteracting-spinless-fermion cluster Green's-function matrix $G_C(p)$ for a cluster of p sites, which we can write as[28]

$$\det G_C(p) = \prod_{l=1}^{p}\lambda_l = \prod_{l=1}^{p}\frac{1}{e^{\varphi_l}+1}, \tag{36}$$

where λ_l are the eigenvalues of the cluster Green's-function matrix $G_C(p)$, and φ_l are the single-particle pseudoenergies of the cluster density matrix ρ_C, for the cluster of p sites in an infinite chain of noninteracting spinless fermions.

For $p\gg 1$, $G_C(p)$ has approximately $\bar{n}p$ eigenvalues which are almost 1, and approximately $(1-\bar{n})p$ eigenvalues which are almost zero. The determinant of $G_C(p)$ is thus determined predominantly by the approximately $(1-\bar{n})p$ eigenvalues which are almost zero. For these λ_l, $e^{\varphi_l}\gg 1$, and thus

$$\det G_C(p) \approx \prod_{\lambda_l\sim 0} e^{-\varphi_l} = \exp\left(-\sum_{l_F}^{l_F+(1-\bar{n})p}\varphi_l\right), \tag{37}$$

where l_F is such that $\varphi_{l_F}=0$. Converting the sum into an integral and using the approximate scaling formula in Ref. 28, we find that

$$\det G_C(p) \approx \exp\left[-p\int_0^{1-\bar{n}} f(\bar{n},x)\,dx\right], \tag{38}$$

i.e., the probability of finding a compact p-bound-pair cluster decays exponentially with p in the limit of $p\gg 1$.

With this simple compact cluster argument, we conclude that the ladder FL correlation $\langle c^{\dagger}_{i,j}c_{i,j+r}\rangle$ decays exponentially with separation r as

$$\langle c^{\dagger}_{i,j}c_{i,j+r}\rangle \sim \exp[-r/\xi(\bar{N}_2)], \tag{39}$$

with a density-dependent correlation length

$$\xi(\bar{N}_2) = \frac{2}{\int_0^{1-\bar{n}(\bar{N}_2)} f(\bar{n}(\bar{N}_2),x)\,dx} \tag{40}$$

in the strong correlated hopping limit. From Ref. 28 we know that the scaling function $f(\bar{n},x)$ depends only very weakly on $\bar{n}$, and thus, at very low ladder densities $\bar{N}_2\to 0$, the correlation length $\xi(\bar{N}_2)$ attains its minimum value of

$$\xi(0) = \frac{2}{\int_0^1 f(0,x)\,dx}, \tag{41}$$

and the FL correlation $\langle c^{\dagger}_{i,j}c_{i,j+r}\rangle$ decays most rapidly in this regime of $\bar{N}_2\to 0$. This is expected physically, since a long cluster of occupied sites is very unlikely to occur at very low densities, with or without quantum correlations.

In the regime of $\bar{N}_2 \to \frac{1}{2}$, we find $\bar{n} \to 1$, and thus the correlation length $\xi(\bar{N})$ diverges according to Eq. (40). This diverging correlation length tells us nothing about the amplitude of the FL correlation. Indeed, when the ladder becomes half-filled, the two degenerate ground states are inert bound-pair solids. Each of the half-filled-ladder ground-state wave functions consists of a single configuration whereby all available plaquettes are occupied by a bound pair and it is not possible to annihilate a spinless fermion at the $(j+r)$th rung and create another at the jth rung. The FL correlation $\langle c_{i,j}^{\dagger} c_{i,j+r} \rangle$ is thus strictly zero in this half-filled-ladder limit.

C. Two-leg limit

This subsection concerns the ground state in the two-leg limit $t_{\perp} \ll t_{\parallel}$, $t'=0$. Based on energetic considerations, we argue in Sec. IV C 1 that there will be two degenerate ground states, within which successive spinless fermions are on alternate legs of the ladder. We call these the *staggered ground states* and write their wave functions in terms of the Fermi-sea ground-state wave function with the help of a *staggered map* between ladder configurations and ordinary chain configurations. We then calculate various ground-state correlations in Secs. IV C 2–IV C 4, where we show that the nonvanishing FL correlations decay exponentially with distance, governed by a density-dependent correlation length, while the CDW and SC correlations decay with distance as power laws. We find in this two-leg limit that the antisymmetric CDW correlation dominates at large distances.

1. Ground states

In the limit of $t_{\perp} \to 0$, each spinless fermion on the two-legged ladder carries a permanent leg index, and thus the number of spinless fermions P_i on leg i are good quantum numbers. Furthermore, successive spinless fermions along the ladder cannot move past each other, even if they are on different legs, because of the infinite nearest-neighbor repulsion acting across the rungs. Consequently, the Hilbert space of the P-spinless-fermion problem breaks up into many independent sectors, each with a fixed sequence of leg indices. The P-spinless-fermion problem in one such sector is therefore an independent problem from that of another P-spinless-fermion sector. Noting that the closest approach between two particles on the same leg is $r=2$, whereas that between two particles on different legs is $r=1$, we invoke the same "particle-in-a-box" argument used for the paired limit in Sec. IV B 1 to find the ground state for P spinless fermion on a ladder of even length L to be in a *staggered* sector, where successive particles are on different legs. There are two such sectors in a ladder with open boundary conditions, which we call sector 1 when the first fermion (from the left) is on leg 1, or sector 2 when it is on leg 2.

Evidently this is a twofold symmetry breaking. (The broken symmetry is that of reflecting the configuration about the ladder axis, which is a valid symmetry within the staggered sector.) This state has a form of long-range order, in that the flavor alternates; however, that cannot be represented by any local order parameter but only by a "string" order parameter.[30]

Let us write $|\Psi_1\rangle$ and $|\Psi_2\rangle$ for the ground states in sectors 1 and 2, respectively. A staggered configuration of P ladder spinless fermions in sector 1 can be mapped to a chain of P noninteracting spinless fermions using the staggered map

$$c_{1,j_1}^{\dagger} c_{2,j_2}^{\dagger} \cdots c_{1,j_{P-1}}^{\dagger} c_{2,j_P}^{\dagger} |0\rangle_{\text{ladder}} \mapsto c_{j_1}^{\dagger} c_{j_2}^{\dagger} \cdots c_{j_{P-1}}^{\dagger} c_{j_P}^{\dagger} |0\rangle_{\text{chain}}. \tag{42}$$

Using the same formula, with an exchange of leg index $1 \leftrightarrow 2$, configurations in sector 2 are similarly mapped.[31]

Because the staggered map maps a ladder with density $\bar{N}_2$ onto a ordinary chain with density $\bar{n}=2\bar{N}_2$, we want a ladder observable O_{ladder} and its corresponding chain observable O_{chain} to be such that

$$\langle O_{\text{ladder}} \rangle_{\text{ladder}} = \frac{1}{2} \langle O_{\text{chain}} \rangle_{\text{chain}}. \tag{43}$$

This is analogous to Eq. (14), which we derived when we map from an excluded chain to an ordinary chain. We use the subscripts "ladder" and "chain" just this once to distinguish between ladder and chain expectations. This notation is cumbersome, so we will not use it again. Whether an expectation is a ladder expectation or a chain expectation should be clear from the context.

2. FL correlations: Exponential decay

Having solved the staggered ground states in terms of the one-dimensional Fermi sea, we calculate the FL, CDW, and SC correlations. There are four FL correlations at range r, $\langle c_{1,j}^{\dagger} c_{j+r} \rangle$, $\langle c_{1,j}^{\dagger} c_{2,j+r} \rangle$, $\langle c_{2,j}^{\dagger} c_{1,j+r} \rangle$, and $\langle c_{2,j}^{\dagger} c_{2,j+r} \rangle$. From the staggered nature of $|\Psi_{\pm}\rangle$, we know that

$$\langle c_{1,j}^{\dagger} c_{1,j+r} \rangle = \langle c_{2,j}^{\dagger} c_{2,j+r} \rangle,$$

$$\langle c_{1,j}^{\dagger} c_{2,j+r} \rangle = \langle c_{2,j}^{\dagger} c_{1,j+r} \rangle \tag{44}$$

in both ground states. The interleg FL correlations vanish, i.e.,

$$\langle c_{1,j}^{\dagger} c_{2,j+r} \rangle = 0 = \langle c_{2,j}^{\dagger} c_{1,j+r} \rangle, \tag{45}$$

because annihilating a particle on one leg and creating a particle on the other leg disrupts the stagger configuration.

The intraleg FL correlation $\langle c_{i,j}^{\dagger} c_{i,j+r} \rangle$, which is nonzero, receives contributions only from initial and final staggered configurations in which there are no intervening particles between rungs j and $j+r$, for example, those shown in Fig. 14. This tells us that

$$\langle c_{i,j}^{\dagger} c_{i,j+r} \rangle = \frac{1}{2} \langle c_j^{\dagger} (\mathbb{1} - n_{j+1}) \cdots (\mathbb{1} - n_{j+r-1}) c_{j+r} \rangle, \tag{46}$$

when we map the ladder model to the chain model. This correlation is evaluated numerically, and shown in Fig. 15, where we see the staggered ground-state FL correlations decaying exponentially with separation r. This asymptotic behavior can again be understood using a constrained probabilities argument similar to that used in Sec. IV B 2, except that instead of a compact cluster, the relevant probability $P(r)$ is that of finding a gap at least r in length within the one-dimensional Fermi-sea ground state.

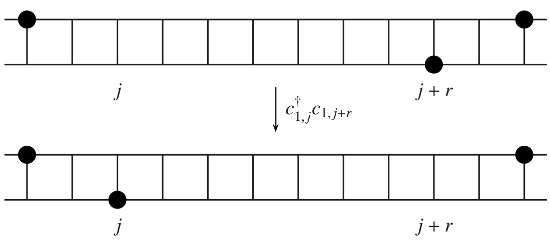

FIG. 14. Annihilation of a spinless fermion at site $(1,j+r)$, followed by creation of a spinless fermion at site $(1,j)$, within a staggered ground-state configuration leads to a staggered ground-state configuration, when there are no intervening particles between rungs j and $j+r$.

Applying a restricted probability argument similar to the one outlined in Sec. IV B 3, we know this probability is simply the zero-particle weight

$$P(r) = w_0 = \det[\mathbb{1} - G_C(r)] \tag{47}$$

of the density matrix of a cluster of r contiguous sites in the chain of noninteracting spinless fermions. For $r \gg 1$, the cluster Green's-function matrix $G_C(r)$ has approximately $(1-\bar{n})r$ eigenvalues which are almost 0 and $\bar{n}r$ eigenvalues which are almost 1. The determinant of $\mathbb{1}-G_C(r)$ is thus essentially determined by the approximately $\bar{n}r$ eigenvalues which are almost 1. Using this fact, we calculate the asymptotic form of $P(s)$ to be

$$P(r) \approx \exp\left\{-r\int_0^{\bar{n}} f(1-\bar{n},x)dx\right\}, \tag{48}$$

where $f(\bar{n},x)$ is the universal scaling function identified in Ref. 28. Equation (48) explains the observed exponential decay of $\langle c^\dagger_{i,j}c_{i,j+r}\rangle$ in Fig. 15. We note further that as $\bar{n}\to 1$ (or equivalently, $\bar{N}_2 \to \frac{1}{2}$), the FL correlations decay fastest exponentially, whereas as $\bar{n}\to 0$ (equivalent to $\bar{N}_2\to 0$), the exponential decay is the slowest. We expect these behaviors physically because it is more likely to find a long empty cluster when the density is low and less likely to find a long empty cluster when the ladder is close to half-filled.

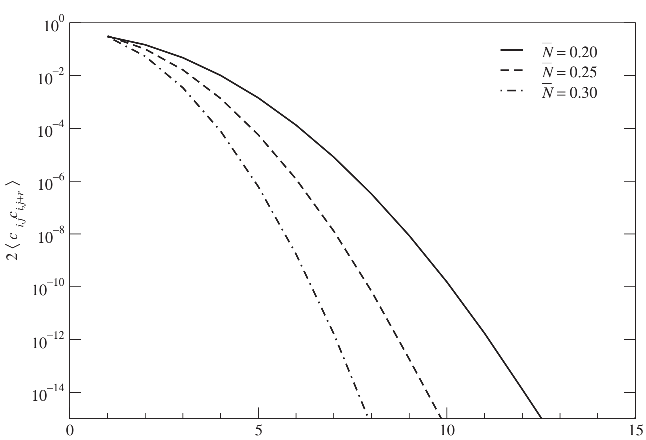

FIG. 15. The infinite-ladder FL correlations $\langle c^\dagger_{i,j}c_{i,j+r}\rangle$, $i=1,2$, as a function of the separation $1\le r\le 15$ for ladder densities $\bar{N}_2 = 0.20$, 0.25, and 0.30, in the two-leg limit $t_\perp/t_\parallel \to 0$, $t'=0$.

3. CDW correlations

Next, we calculate the CDW correlations for which the four simplest at separation r are

$$\langle c^\dagger_{1,j}c_{1,j}c^\dagger_{1,j+r}c_{1,j+r}\rangle = \langle c^\dagger_{2,j}c_{2,j}c^\dagger_{2,j+r}c_{2,j+r}\rangle,$$

$$\langle c^\dagger_{1,j}c_{1,j}c^\dagger_{2,j+r}c_{2,j+r}\rangle = \langle c^\dagger_{2,j}c_{2,j}c^\dagger_{1,j+r}c_{1,j+r}\rangle. \tag{49}$$

Because of the staggered nature of the ground states, configurations making nonzero contributions to $\langle n_{i,j}n_{i,j+r}\rangle$ are those which map to noninteracting spinless fermion configurations in which the sites j and $j+r$ are occupied with an odd number of intervening particles between them. Similarly, configurations making nonzero contributions to $\langle n_{i,j}n_{i',j+r}\rangle$, $i\neq i'$ are those which map to noninteracting spinless fermions in which the sites j and $j+r$ are occupied with an even number of intervening particles between them.

Defining the density operators

$$n_{\pm,j} \equiv n_{1,j} \pm n_{2,j}, \tag{50}$$

which are symmetric and antisymmetric with respect to reflection along the ladder axis, we find that

$$\langle n_{+,j}n_{-,j+r}\rangle = 0 = \langle n_{-,j}n_{+,j+r}\rangle \tag{51}$$

and

$$\langle n_{+,j}n_{+,j+r}\rangle = \langle n_j n_{j+r}\rangle = \Sigma_+(r), \tag{52}$$

which we call the CDW+ *correlation*. This is identical to the CDW correlation of the one-dimensional Fermi sea, which we know decays as an oscillatory power law

$$\langle n_{+,j}n_{+,j+r}\rangle - \langle n_{+,j}\rangle\langle n_{+,j+r}\rangle \sim r^{-2}\cos(2k_F r). \tag{53}$$

There is also the CDW− *correlation*,

$$\langle n_{-,j}n_{-,j+r}\rangle = 2(\langle n_{1,j}n_{1,j+r}\rangle - \langle n_{1,j}n_{2,j+r}\rangle) = \Sigma_-(r), \tag{54}$$

associated with $n_{-,j}$. This is identical to the subtracted CDW− correlation, since $\langle n_{-,j}\rangle = \langle n_{1,j}-n_{2,j}\rangle = 0$. Evaluating this expectation numerically, we find that at all ladder densities $\bar{N}_2$, $\Sigma_-(\bar{N}_2,r)$ oscillates about a zero average with wave vector $2k_F$ and a decaying amplitude. A preliminary unrestricted nonlinear curve fitting to the asymptotic form

$$\Sigma_-(\bar{N}_2,r) = B_0 r^{-\beta_0} + B_1 r^{-\beta_1}\cos(2k_F r + \theta_1), \tag{55}$$

where $B_1 r^{-\beta_1}\cos(2k_F r+\theta_1)$ is the leading asymptotic behavior and $B_0 r^{-\beta_0}$ is a correction term, suggests that the leading correlation exponent may actually be universal taking on the value $\beta_1 = \frac{1}{2}$. Further nonlinear curve fitting, restricting $\beta_1 = \frac{1}{2}$, tells us that only the parameters B_1 and θ_1 of the leading asymptotic term can be reliably determined. These are shown in Fig. 16. From this restricted curve fit, it appears that the phase shift might also be universal taking on value $\theta_1 = \pi$.

4. SC correlations

The simplest SC correlations at separation r are

$$\langle c^\dagger_{1,j}c^\dagger_{2,j+1}c_{1,j+r}c_{2,j+r+1}\rangle = \langle c^\dagger_{2,j}c^\dagger_{1,j+1}c_{2,j+r}c_{1,j+r+1}\rangle,$$

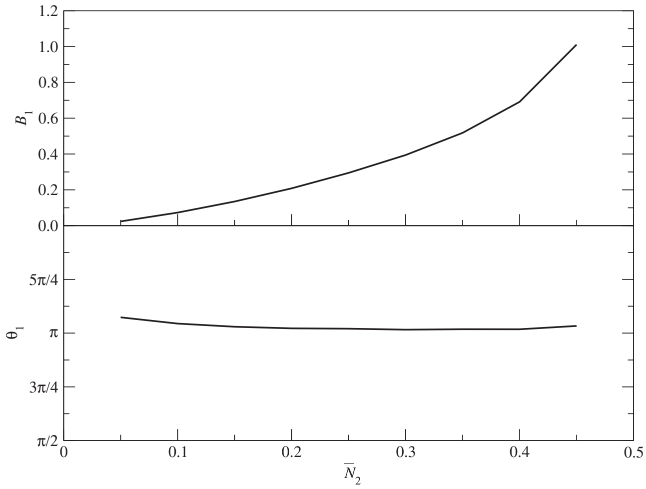

FIG. 16. Plot of the fitted amplitude B_1 (top) and fitted phase shift θ_1 (bottom) of the leading oscillatory power-law decay, as functions of the ladder density $\bar{N}_2$, for the CDW− correlation in the staggered ground state of the ladder model, in the two-leg limit $t_\perp/t_\parallel \to 0$, $t'=0$.

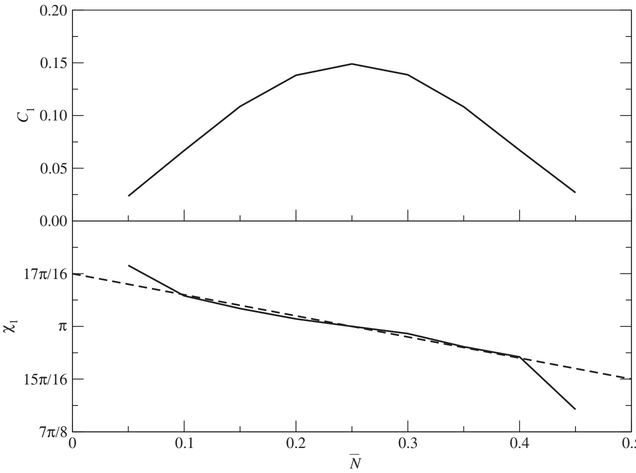

FIG. 17. Plot of the fitted amplitude C_1 (top) and fitted phase shift χ_1 (bottom) of the leading oscillatory power-law decay, as functions of the ladder density $\bar{N}_2$, for the SC− correlation in the staggered ground state of the ladder model, in the two-leg limit $t_\perp/t_\parallel \to 0$, $t'=0$.

$$\langle c_{1,j}^\dagger c_{2,j+1}^\dagger c_{2,j+r} c_{1,j+r+1} \rangle = \langle c_{2,j}^\dagger c_{1,j+1}^\dagger c_{1,j+r} c_{2,j+r+1} \rangle. \tag{56}$$

Correlations of the type $\langle c_{i,j}^\dagger c_{i',j+1}^\dagger c_{i,j+r} c_{i',j+r+1} \rangle$ receive nonzero contributions from configurations containing an even number of intervening particles between rungs $j+1$ and $j+r$, whereas correlations of the type $\langle c_{i,j}^\dagger c_{i',j+1}^\dagger c_{i',j+r} c_{i,j+r+1} \rangle$ receive nonzero contributions from configurations containing an odd number of intervening particles between rungs $j+1$ and $j+r$. Defining the paired operators

$$\Delta_{\pm,j}^\dagger = \frac{1}{\sqrt{2}} (c_{1,j}^\dagger c_{2,j+1}^\dagger \pm c_{1,j+1}^\dagger c_{2,j}^\dagger), \tag{57}$$

which are symmetric and antisymmetric with respect to reflection about the ladder axis, we find that

$$\langle \Delta_{+,j}^\dagger \Delta_{-,j+r} \rangle = 0 = \langle \Delta_{-,j}^\dagger \Delta_{+,j+r} \rangle, \tag{58}$$

and that the SC+ correlation

$$\langle \Delta_{+,j}^\dagger \Delta_{+,j+r} \rangle = \langle c_j^\dagger c_{j+1}^\dagger c_{j+r} c_{j+r+1} \rangle = \Pi_+(r) \sim r^{-2} \tag{59}$$

at large separations.

The SC− correlation

$$\langle \Delta_{-,j}^\dagger \Delta_{-,j+r} \rangle = \Pi_-(r) \tag{60}$$

must be evaluated numerically. We find that, just like $\Sigma_-(r)$, $\Pi_-(r)$ oscillates about a zero average with wave vector $2k_F$ and a rapidly decaying amplitude. To improve the quality of the nonlinear curve fitting, we fit $r^2\Pi_-(r)$ to the asymptotic form

$$r^2\Pi_-(r) = C_0 r^{2-\beta_0} + C_1 r^{2-\beta_1} \cos(2k_F r + \chi_1), \tag{61}$$

where $C_1 r^{2-\beta_1} \cos(2k_F r + \chi_1)$ is the leading asymptotic behavior, while $C_0 r^{2-\beta_0}$ is a correction term. A preliminary unrestricted fit suggests that the leading correlation exponent is universal and takes on value $\beta_1 = \frac{5}{2}$. Further restricted nonlinear curve fitting tells us that only the parameters C_1 and χ_1 can be reliably determined. These are shown in Fig. 17, where we see that the amplitude C_1 exhibits symmetry about quarter filling, which is a kind of particle-hole symmetry, and that the phase shift $\chi_1 = \pi[1 + \frac{1}{4}(\frac{1}{4} - \bar{N}_2)]$ is nonuniversal but depends linearly on the density $\bar{N}_2$.

D. Rung-fermion limit

In this subsection, we look at the rung-fermion limit $t_\perp \gg t_\parallel$, $t'=0$. We argue in Sec. IV D 1 that in this limit, each spinless fermion spends most of its time hopping back and forth along the rung it is on and only very rarely hops along the legs to an adjacent rung. Therefore, each spinless fermion will be in a quantum state very close to the symmetric eigenstate of one rung and we can think of the ladder of spinless fermions with density $\bar{N}_2$ in this limit as essentially an excluded chain of fermions with density $\bar{N} = 2\bar{N}_2$. For $\bar{N}_2 < \frac{1}{4}$, the ground state of this excluded chain of rung fermions has been solved in Sec. II A. The FL, CDW, and SC correlations have also been calculated in Sec. III so we will not repeat them here.

At $\bar{N}_2 = \frac{1}{4}$, the ground state is a "dynamic solid" phase, in which rung fermions occupy either all the even rungs, or all the odd rungs, and cannot hop along the legs to adjacent rungs because of the infinite nearest-neighbor repulsion between them. For $\bar{N}_2 > \frac{1}{4}$, we describe in Sec. IV D 2 how the system will phase separate into a high-density inert solid phase, in which spinless fermions cannot hop at all, and the lower-density dynamic solid phase. In this phase separation regime, the FL, CDW, and SC correlations cannot be calculated.

1. Ground states

In the limit of $t_\parallel / t_\perp \to 0$, a spinless fermion spends most of its time hopping back and forth along a rung and only very rarely hops along the leg to an adjacent rung where it will spend a lot of time hopping back and forth before hopping along the leg again. Because of this long dwell time on a rung, the spinless fermion is very nearly in the rung ground state

$$|+,j\rangle = \frac{1}{\sqrt{2}}(c_{1,j}^{\dagger} + c_{2,j}^{\dagger})|0\rangle = C_j^{\dagger}|0\rangle. \tag{62}$$

Let us call a spinless fermion in the rung ground state a *rung fermion* in short. Rung fermions inherit the infinite nearest-neighbor repulsion of the bare spinless fermions, and therefore two rung-fermions in adjacent rungs experience infinite nearest-neighbor repulsion as well. With this insight, we find that the full many-body problem of spinless fermions with infinite nearest-neighbor repulsion on the two-legged ladder with density $\bar{N}_2$ reduces to the problem of an excluded chain with density $\bar{N}=2\bar{N}_2$ of spinless rung fermions.

The latter problem was solved in Secs. II and III for excluded chain densities $\bar{N}<\frac{1}{2}$. In the special case of quarter filling on the ladder, $\bar{N}_2=\frac{1}{4}$, spinless fermions occupy alternate rungs. These are free to hop along the rungs that they reside on, but cannot hop along the legs, for nonvanishing values of $t_{\parallel}/t_{\perp}$. Even virtual processes in which a spinless fermion on rung j hops along the leg to an adjacent rung and back are essentially forbidden by the infinite nearest-neighbor repulsion, because such virtual processes, which has a time scale of $O(1/t_{\parallel})$, would not be complete when the spinless fermion on the next-nearest-neighbor rung hops across the rung, which occurs on a time scale of $O(1/t_{\perp})$. Virtual processes such as these only become energetically feasible when the two time scales become comparable, i.e., when $t_{\parallel}\lesssim t_{\perp}$. Therefore, over a wide range of anisotropies $t_{\parallel}/t_{\perp}$, the spinless fermions in the quarter-filled ladder with $t'=0$ can hop back and forth along the rungs they are on but cannot hop to the neighboring rungs. This gives rise to a symmetry breaking, where the spinless fermions are either all on the even rungs or they are all on the odd rungs. Because translational symmetry along the ladder axis is broken in the quarter-filled ladder ground states, we think of these as dynamic solids, since the constituent spinless fermions are constantly hopping back and forth along the rungs. In this limit, the only nonvanishing correlation is the rung-fermion CDW correlation

$$\langle N_j N_{j+r}\rangle = \begin{cases} \frac{1}{2}, & r \text{ even} \\ 0, & r \text{ odd}, \end{cases} \tag{63}$$

i.e., there is true long-range order in the quarter-filled ladder ground state in the limit of $t_{\perp}\gg t_{\parallel}$, $t'=0$.

2. Phase separation

In this rung-fermion limit $t_{\perp}\gg t_{\parallel}$, $t'=0$, the system phase separates for ladder densities $\bar{N}_2>\frac{1}{4}$. As shown in Fig. 18, when the ladder is above quarter filling, some of the spinless fermions will go into a high-density inert solid phase with density $\bar{N}_2=\frac{1}{2}$, where spinless fermions are arranged in a staggered array and therefore cannot hop at all. These spinless fermions contribute nothing to the ground-state energy. If $t_{\parallel}$ is comparable to $t_{\perp}$, the rest of the spinless fermions will go into a fluid phase, whose density is $\bar{N}_2<\frac{1}{4}$. These spinless fermions are free to hop back and forth on the rungs they are on, and occasionally to the neighboring rungs, when permitted by nearest-neighbor exclusion. These contribute a density-dependent total kinetic energy to the ground-state energy. The ground-state composition depends on whether the kinetic energy gained per particle, by removing a spinless fermion from the solid phase and adding it to the fluid phase, outweighs the decrease in kinetic energy per particle that results from the fluid becoming more congested.

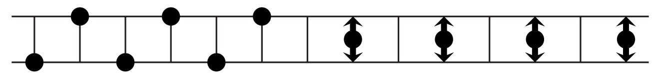

FIG. 18. Phase separation of a greater-than-quarter-filled ladder of spinless fermions with infinite nearest-neighbor repulsion into a high-density inert solid phase (immobile spinless fermions) with $\bar{N}_2=\frac{1}{2}$ and a low-density fluid phase (mobile spinless fermions shown with arrows) with $\bar{N}_2=\frac{1}{4}$ in the rung-fermion limit $t_{\perp}\gg t_{\parallel}$, $t'=0$.

When $t_{\perp}$ becomes large compared to $t_{\parallel}$, which is the limit we are interested in, it becomes energetically favorable, always, to remove one spinless fermion from the inert solid phase, and add it to the fluid phase, if its density is $\bar{N}_2<\frac{1}{4}$. This is because the kinetic energy penalty to make the fluid becoming more congested, which is of $O(t_{\parallel})$, is more than compensated for by the kinetic energy gain of $t_{\perp}$ for an extra spinless fermion freed to hop back and forth along a rung. Iterating this argument, we find then that, for $t_{\perp}\gg t_{\parallel}$, and the overall density $\bar{N}_2>\frac{1}{4}$, the system will phase separate into an inert solid phase with density $\bar{N}_2=\frac{1}{2}$, and a dynamic solid phase with density $\bar{N}_2=\frac{1}{4}$. For example, if the overall density is $\bar{N}_2=\frac{1}{3}>\frac{1}{4}$, we will find that $\frac{1}{3}$ of the total number of spinless fermions will be in the inert solid phase, while the other $\frac{2}{3}$ of the total number of spinless fermions will be in the dynamic solid phase.

V. SUMMARY AND DISCUSSIONS

In this paper, we established a one-to-one correspondence between P-particle configurations on the excluded chain and P-particle configurations on the ordinary chain using the right-exclusion map. We then showed that the Hamiltonian matrices of the models given in Eqs. (1) and (3) are identical, therefore, solving for the ground states of the former in terms of those of the latter. These results were obtained for finite chains subject to open boundary conditions but continue to hold for infinite chains.

Based on this one-to-one correspondence between ground states, we showed that the ground-state expectation $\langle O\rangle$ of an excluded chain observable O can be evaluated using Eq. (14) in terms of the ground-state expectation $\langle O'\rangle$ of a carefully chosen corresponding observable O' on the ordinary chain. We then developed the method of intervening-particle expansion to write the ground-state expectation $\langle O_j O_{j+r}\rangle$ of a product of local excluded chain operators O_j and O_{j+r}, first as a sum over excluded chain expectations $\langle O_j O_p O_{j+r}\rangle$ conditioned on the occupations of the sites between j and $j+r$, and then as a sum over the corresponding ordinary chain expectations $\langle O'_j O'_p O'_{j+r-p}\rangle$.

Using these analytical results from Sec. II, we calculated the FL, CDW, and SC correlations of the excluded chains of hardcore bosons and spinless fermions in Sec. III. Based on nonlinear curve fits of the numerically evaluated correlations, to reasonable asymptotic forms, we find all three types of correlations decaying with separation r as power laws for hardcore bosons as well as for spinless fermions. More interestingly, we find for both hardcore bosons and spinless fermions a universal exponent $\gamma_1=\frac{7}{4}$ for the oscillatory power-law decay of the SC correlation, but a nonuniversal, density-dependent, exponent $\beta_1=\frac{1}{2}+\frac{5}{2}(\frac{1}{2}-\bar{N})$ for the oscillatory power-law decay of the CDW correlation. Also, the leading asymptotic behavior for the hardcore boson FL correlation was found to a nonoscillating power-law decay with universal exponent $\alpha_0=\frac{1}{2}$, while that for the spinless fermion FL correlation was found to be oscillations in a power-law envelope with a nonuniversal exponent that approaches $\alpha_1=1$ as $\bar{N}\to 0$ and $\alpha_1=\frac{1}{4}$ as $\bar{N}\to\frac{1}{2}$.

We then analyzed our spinless-fermion ladder model, Eq. (2), in Sec. IV. This ladder model can be solved exactly in three limiting cases: (i) the paired limit $t'\gg t_\parallel$, $t_\perp$; (ii) the two-leg limit $t_\perp \ll t_\parallel$, $t'=0$; and (iii) the rung-fermion limit $t_\perp \gg t_\parallel$, $t'=0$. In the paired limit, which we solved in Sec. IV B, spinless fermions form correlated-hopping bound pairs, and so the ladder model can be mapped to the excluded chain of hardcore bosons. The ground state of this latter model was solved exactly in Sec. II and its ground-state correlations calculated in Sec. III. By reinterpreting the excluded chain correlations in ladder terms, we realized that ladder SC correlations dominate at large distances over ladder CDW correlations, both of which decay as power laws with separation with leading exponents $\gamma_0=\frac{1}{2}$ and $\beta_1=\frac{1}{2}+\frac{5}{2}(\frac{1}{2}-\bar{N}_2)$, respectively, $\bar{N}_2$ being the ladder density. We also showed using a restricted probabilities argument that ladder FL correlations decay exponentially with separation, with a density-dependent correlation length.

Next, in the two-leg limit, which we solved in Sec. IV C, we argued based on a particle-in-a-box picture that successive spinless fermions in the twofold degenerate *staggered ground states* occupy different legs of the ladder. We write these ground states exactly in terms of the one-dimensional Fermi sea in Sec. IV C 1 before calculating correlations in Sec. IV C 2. We found using a different restricted probabilities argument that FL correlations decay exponentially with separation, with a density-dependent correlation length. CDW and SC correlations symmetric (antisymmetric) with respect to a reflection about the ladder axis decay as power laws with universal leading exponents $\beta_1=2(\frac{1}{2})$ and $\gamma_1=2(\frac{5}{2})$, respectively.

Finally, in the rung-fermion limit, we mapped the ladder model to an excluded chain of spinless fermions in Sec. IV D. Since we have already solved this latter model in Sec. II and calculated its ground-state correlations in Sec. III below half-filling (which corresponds to quarter filling on the ladder), we discussed the phase separations that occur on ladders with greater than quarter filling in Sec. IV D 1. Correlation exponents obtained for the three limiting cases of our ladder model Eq. (2) as well as those for the excluded chains of hardcore bosons and spinless fermions are summarized in Table I. A recent density-matrix renormalization group study[32] found the same qualitative features in the correlation exponents in the paired limit. However, they see more filling-dependent exponents with numerical values different from ours.

In this study, we find the emergence of surprising universal correlation exponents. In the Luttinger liquid paradigm, all correlation exponents can be written in terms of the exponents[33–35]

$$\gamma_\rho=\frac{1}{8}(K_\rho+K_\rho^{-1}-2),$$

$$\gamma_\sigma=\frac{1}{8}(K_\sigma+K_\sigma^{-1}-2) \tag{64}$$

appearing in the quantum-mechanical propagator, also called the (equal-time) two-point function

$$\begin{aligned}G(r) &\sim A_1 r^{-\alpha}\cos k_F r = A_1 r^{-[1+2(\gamma_\rho+\gamma_\sigma)]}\cos k_F r\\ &= A_1 r^{-1/4[(K_\rho+K_\rho^{-1})+(K_\sigma+K_\sigma^{-1})]}\cos k_F r.\end{aligned} \tag{65}$$

The parameters K_ρ and K_σ depend generically on the filling fraction and the interaction strength, and thus all correlation exponents are nonuniversal. In particular, various theoretical approaches (see review by Sólyom)[33] tell us that the CDW, spin-density wave, singlet superconductivity, and triplet superconductivity correlations decay as power laws

$$\langle n(0)n(r)\rangle \sim \frac{K_\rho}{\pi^2 r^2}+B_2 r^{-K_\rho-K_\sigma}\cos 2k_F r + B_4 r^{-4K_\rho}\cos 4k_F r, \tag{66a}$$

$$\langle\sigma_x(0)\sigma_x(r)\rangle=\langle\sigma_y(0)\sigma_y(r)\rangle \sim \frac{D_{0,xy}}{r^2}+D_{2,xy}r^{-K_\rho-K_\sigma^{-1}}\cos 2k_F r, \tag{66b}$$

$$\langle\sigma_z(0)\sigma_z(r)\rangle \sim \frac{D_{0,z}}{r^2}+D_{2,z}r^{-K_\rho-K_\sigma}\cos 2k_F r, \tag{66c}$$

$$\langle\Delta_{0,0}^\dagger(0)\Delta_{0,0}(r)\rangle=\langle\Delta_{1,0}^\dagger(0)\Delta_{1,0}(r)\rangle \sim C_0 r^{-K_\rho^{-1}-K_\sigma}, \tag{66d}$$

$$\langle\Delta_{1,\pm1}^\dagger(0)\Delta_{1,\pm1}(r)\rangle \sim C'_0 r^{-K_\rho^{-1}-K_\sigma^{-1}} \tag{66e}$$

in a Tomonaga-Luttinger liquid.

When the chain of interacting spinfull fermions is spin-rotation invariant (for example, in the absence of an external magnetic field), the spin stiffness constant must be $K_\sigma=1$ and the ground-state properties become completely determined by the single nontrivial Luttinger parameter K_ρ. The spinfull power laws thus become

TABLE I. A summary of the leading correlation exponents and wave vectors of various correlation functions that decay as power laws in the (i) paired limit $t' \gg t_\parallel, t_\perp$; (ii) two-leg limit $t_\perp \ll t_\parallel$, $t'=0$; and (iii) rung-fermion limit $t_\perp \gg t_\parallel$, $t'=0$. The wave vector k of the leading terms in the correlation functions are reported in terms of $k_F=\pi\bar{N}_1$, where $0\le\bar{N}_1\le\frac{1}{2}$ is the excluded chain density. The suffixes π and $\pm$ indicate further symmetries possible in the ladder model.

Model	Correlation function	Correlation exponent	Wave vector
Hardcore boson	FL	$\frac{1}{2}$	0
	CDW	$\frac{1}{2}+\frac{5}{2}(\frac{1}{2}-\bar{N}_1)$	$2k_F$
	SC	$\frac{7}{4}$	0
Spinless fermion	FL	$1\rightarrow\frac{1}{4}$	k_F
	CDW	$\frac{1}{2}+\frac{5}{2}(\frac{1}{2}-\bar{N}_1)$	$2k_F$
	SC	$\frac{7}{4}$	0
$t' \gg t_\parallel, t_\perp$	CDW-π	$\frac{1}{2}+\frac{5}{2}(\frac{1}{2}-\bar{N}_1)$	$2k_F$
		2	0
	SC	$\frac{1}{2}$	0
		$\frac{3}{2}\rightarrow\frac{1}{2}$	$2k_F$
$t_\perp \ll t_\parallel$, $t'=0$	CDW+	2	0
		2	$2k_F$
	CDW−	$\frac{1}{2}$	$2k_F$
		2	0
	SC+	2	0
		2	$2k_F$
	SC−	$\frac{5}{2}$	$2k_F$
		4	0
$t_\perp \gg t_\parallel$, $t'=0$	FL	$1\rightarrow\frac{1}{4}$	k_F
	CDW	$\frac{1}{2}+\frac{5}{2}(\frac{1}{2}-\bar{N}_1)$	$2k_F$
		2	0
	SC	$\frac{7}{4}$	0

$$G(r) \sim A_1 r^{-1/4(K_\rho+K_\rho^{-1}+2)} \cos k_F r, \tag{67a}$$

$$\langle n(0)n(r)\rangle \sim \frac{K_\rho}{\pi r^2} + B_2 r^{-K_\rho-1}\cos 2k_F r + B_4 r^{-4K_\rho}\cos 4k_F r, \tag{67b}$$

$$\langle \boldsymbol{\sigma}(0)\cdot\boldsymbol{\sigma}(r)\rangle \sim \frac{1}{\pi r^2} + D_2 r^{-K_\rho-1}\cos 2k_F r, \tag{67c}$$

$$\langle \Delta_0^\dagger(0)\Delta_0(r)\rangle = \langle \Delta_1^\dagger(0)\Delta_1(r)\rangle \sim C_0 r^{-K_\rho^{-1}-1}. \tag{67d}$$

For spinless fermions, there is only one independent stiffness constant $K_\rho=K_\sigma=K$,[35,36] so that the spinfull power laws which have proper spinless analogs are

$$G(r) \sim A_1 r^{-1/2(K+K^{-1})} \cos k_F r, \tag{68a}$$

$$\langle n(0)n(r)\rangle \sim \frac{K}{\pi r^2} + B_2 r^{-2K}\cos 2k_F r + B_4 r^{-4K}\cos 4k_F r. \tag{68b}$$

In the Luttinger liquid paradigm, universal correlation exponents arise in the special cases of the Fermi liquid, where we have $K_\rho=1$ [consequently, the two-point function decays as $G(r)\sim r^{-1}$, while the CDW and SC correlations both decay as r^{-2}], as well as the one-dimensional Hubbard model with infinite onsite interaction, where we have $K_\rho=\frac{1}{2}$.[37] However, we found a number of universal correlations exponents that are different from those expected of a free Fermi sea, which our exact solutions are mapped to. Furthermore, the correlation exponents α, β, and γ of the FL, CDW, and SC correlations ought to obey definite relations in a Luttinger liquid, because they can all be written in terms of a single Luttinger parameter K. Again, the universal and nonuniversal correlation exponents we find in our exact solutions do not obey these relations. These observations bring us to the paper by Efetov and Larkin, who first calculated the universal FL correlation exponent for an ordinary chain of hardcore bosons to be $\alpha=\frac{1}{2}$.[16] If we accept for the moment that the Luttinger paradigm is correct and that universal correlation exponents are those of the Fermi-liquid fixed point then we are led to the conclusion that $\alpha=\frac{1}{2}$ must be a correlation exponent of the Fermi liquid. Clearly, this exponent does not belong to the FL correlation (which should be $\alpha=1$), so what correlation does it belong to?

In the seminal paper by Jordan and Wigner, the ordinary chain of hardcore bosons is mapped to the ordinary chain of spinless fermions using the Jordan-Wigner transformation (see the Appendix). In this transformation, the hardcore boson point operators $b_j^\dagger$ and b_{j+r} are each mapped to spinless fermion *string operators* $c_j^\dagger\prod_{i<j}(-1)^{n_i}$ and $\prod_{i<j+r}(-1)^{n_i}c_{j+r}$, respectively. The FL correlation $\langle b_j^\dagger b_{j+r}\rangle$ between two hardcore boson point operators thus becomes the expectation $\langle c_j^\dagger\prod_{i=j+1}^{i=j+r-1}(-1)^{n_i}c_{j+r}\rangle$ of the string operator $c_j^\dagger\prod_{i=j+1}^{i=j+r-1}(-1)^{n_i}c_{j+r}$, which Efetov and Larkin found to decay with separation r as $r^{-1/2}$. String correlations such as this have never been systematically studied. One reason for this lack of interest is that typical string correlations, which receive contributions only from restricted classes of configurations, decay exponentially with r, as we have seen for the FL correlation in the paired limit (Sec. IV B 2) and the two-leg limit (Sec. IV C 2). However, there appear to many string correlations that decay with separation r as power laws. These power-law decays are associated with (quasi-)long-range order that we have not been creative enough to imagine.

In Sec. IV C 1, we found in the two-leg limit that the staggered ground state has long-range order, in that if we know the pth particle is on leg $i=1$, then we know for certain that the $(p+2s)$th particle is on leg $i=1$, and the $(p+2s+1)$th particle is on leg $i=2$, even as $s\to\infty$, and even though we have no idea where these particles are on the ladder. This long-range order is not the usual kind of long-range order, which can be written in terms of the correlation between local order parameters, but is a long-range *string* order. The map from the ordinary chain ground state to the staggered ladder ground state, which is the inverse of the one constructed in Sec. IV C 1, implicitly involves string operators, in that if we take the pth particle in the ordinary ground-state configuration, we will know whether to map it to a particle on leg $i=1$ or leg $i=2$, *after* we know which legs the preceding particles are on. Also, while it is deceptively simple to describe what the string operator in this inverse map does, which is to project out any combination of more than or equal to two consecutive particles on the same leg of the ladder, we know of no compact way to write down the string operator, even in this simple limit, unlike for the case of the Jordan-Wigner string.

What we do know, drawing parallels from the Jordan-Wigner map from hardcore bosons on ordinary chains to noninteracting spinless fermions, is that a string map from one model to another will map some products of local operators to string operators, for example, the hardcore boson $b_j^\dagger b_{j+r}$ to the spinless fermion $c_j^\dagger\prod_{j'=j+1}^{j'=j+r-1}(-1)^{n_{j'}}c_{j+r}$, and other products of local operators to products of local operators, for example, the hardcore boson n_jn_{j+r} to the spinless fermion n_jn_{j+r}. Having understood this, we realized that the CDW+ and SC+ correlations in the staggered ground state get mapped to the correlation of local operators because the string operators involved in the map multiply and cancel each other. On the other hand, when we map the CDW− and SC− staggered ground-state correlations to correlations of a chain of noninteracting spinless fermions, the string operators involved in the map do not cancel each other, and thus the resulting ordinary chain spinless-fermion correlations are string correlations. We also realized that these string correlations are operationally defined by the intervening-particle expansions we used to compute them. Like the familiar Jordan-Wigner string operators, our string operators only depend on the next count of fermions between two points. Since fermions are conserved, that corresponds to field operators in the continuum description—no fermions fall between the cracks in the coarse graining. Therefore, the asymptotic exponents claimed to be universal in this paper are not sensitive to short-range correlations.

Since all the exact solutions we have obtained in this paper can ultimately be mapped to the one-dimensional Fermi sea, we conjecture that all correlation exponents are universal. We claim that: (i) all exponents that are explicitly universal are simple rational polynomials of the single universal spinless Fermi-liquid parameter $K=1$; and (ii) nonuniversal exponents are the result of (under)fitting linear combinations of universal power laws to a single power law. For example, in the two-leg limit, the leading universal exponent $\beta_1=\frac{1}{2}$ of the CDW− correlation in the staggered ground state can be shown using a bosonization calculation of the string correlation it is mapped to, to follow automatically from the universal Fermi-liquid parameter $K=1$.[23] In this same limiting case, the leading universal correlation exponent $\gamma_1=\frac{5}{2}$ of the SC − correlation, which gets mapped to a significantly more complicated string correlation, can conceivably be written as the combination

$$2K+\frac{1}{2K}=\frac{5}{2} \tag{69}$$

of the universal Fermi-liquid parameter $K=1$, even though the bosonized form of this string correlation is not known. For the excluded chain of hardcore bosons or spinless fermions, nonlinear curve fitting of the SC correlation to the sum of one leading power-law decay and one subleading power-law decay leads to weakly nonuniversal correlation exponents for both power laws, whereas a complicated sum of universal power-law decays, Eq. (26), produces a better fit visually. We suspect that good fits can also be obtained, using similar complicated sum of universal power-law decays, for those numerical correlations which we fitted to strongly nonuniversal power laws. Ultimately, the correlation exponents obtained through nonlinear curve fitting must be understood as some form of best estimate or bounds for the true asymptotic behavior, not only for string correlations, but for *all* numerically evaluated correlations. The main difficulty lies with subdominant correlations, the most important of which can be different at different ranges, and how large the amplitudes of these correlations compared to the dominant long-range correlation. Based on our observations, the fits also become apparently less reliable near zero (small fermion density) or maximum filling (small hole density). The small fermion (or hole) density sets a large length scale that we have to cross over to get the asymptotic behavior. These systematic errors at a fixed distance R being nonuniform, and getting worse as we approach zero or maximum filling, affects any methods, e.g., exact diagonalization or quantum Monte Carlo.

Finally, we asked ourselves whether all these string correlations that we have predicted will decay with separation r slower than the two-point function $\langle c_j^\dagger c_{j+r}\rangle \sim r^{-1}$ can be measured in a chain of noninteracting spinless fermions. Since these string operators are nonlocal observables, they do not in general couple to local measurements, so direct experimental measurement would be challenging if not downright impossible. However, we would like to suggest the following possibility: for a given string correlation of the one-dimensional Fermi sea, cook up in the laboratory an experimental system in which the corresponding correlation is a point correlation. If the ground state of the experimental system can be mapped to the one-dimensional Fermi sea, we expect a measurement of the point correlation exponent in the experimental system to be an indirect measurement of the string correlation exponent in the Fermi sea.

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APPENDIX: JORDAN-WIGNER TRANSFORMATION

On a one-dimensional chain, hardcore bosons cannot move past each other, as one boson must first hop on top of the other—a move explicitly forbidden by the hardcore condition—for this to happen. For a different reason (the Pauli exclusion principle), but to the same effect, noninteracting spinless fermions on a one-dimensional chain cannot exchange positions. Therefore, in one dimension, the hardcore-boson and noninteracting-spinless-fermion Hamiltonians are also identical in structure, and thus the ground state of a chain of hardcore bosons is related to the Fermi-sea ground state of a chain of noninteracting spinless fermions in a simple way. A translation machinery exists to map back and forth between these two ground states. This is the Jordan-Wigner transformation[38]

$$b_i = \prod_{j<i} (\mathbb{1} - 2c_j^\dagger c_j) c_i, \quad b_i^\dagger = c_i^\dagger \prod_{j<i} (\mathbb{1} - 2c_j^\dagger c_j), \tag{A1}$$

which maps hardcore bosons to spinless fermions, where the product

$$\prod_{j<i} (\mathbb{1} - 2c_j^\dagger c_j) = \prod_{j<i} (\mathbb{1} - 2n_j) = \prod_{j<i} (-1)^{n_j} \tag{A2}$$

is called the *Jordan-Wigner string*.

In Sec. IV B we saw how pairs of spinless fermions bound by correlated hops in the limit $t' \gg t_\perp$, $t_\parallel$ can be mapped to hardcore bosons with infinite nearest-neighbor repulsion, and then to hardcore bosons using the right-exclusion map described in Sec. II A, and then finally to noninteracting spinless fermions. In Sec. II B, we saw how excluded hardcore-boson expectations are related to appropriately chosen ordinary hardcore-boson expectations. This relation between excluded hardcore-boson expectations and ordinary hardcore-boson expectations will typically involve the intervening-particle expansion Eq. (15). As such, we will encounter hardcore-boson expectations of the form

$$\langle b_i^\dagger (\mathbb{1} - n_{i+1}) \cdots n_{i+l_1} \cdots n_{i+l_p} \cdots (\mathbb{1} - n_{i+r'-1}) b_{i+r'} \rangle, \tag{A3}$$

a lot, where there are p hardcore-boson occupation number operators n_{i+l}, at sites $i+l$, and $r'-p-1$ hardcore-boson operators $(\mathbb{1}-n_{i+l'})$, at sites $i+l'$, between the hardcore-boson operators $b_i^\dagger$ at site i and $b_{i+r'}$ at site $i+r'$.

To evaluate these expectations, we first invoke the Jordan-Wigner transformation (A1) to replace all the hardcore-boson occupation number operators $n_j = b_j^\dagger b_j$ by spinless-fermion occupation number operators $n_j = c_j^\dagger c_j$ in Eq. (A3). Then, to account for the two unpaired hardcore-boson operators at the ends of the hardcore-boson operator product, we write Eq. (A3) as the spinless-fermion expectation

$$\Big\langle c_i^\dagger \prod_{j<i} (\mathbb{1} - 2n_j)(\mathbb{1} - n_{i+1}) \cdots n_{i+l_1} \cdots n_{i+l_p} \cdots (\mathbb{1} - n_{i+r'-1}) \times \prod_{j<i} (\mathbb{1} - 2n_j) \prod_{i\le j<i+r'} (\mathbb{1} - 2n_j) c_{i+r'} \Big\rangle. \tag{A4}$$

Noting that all Jordan-Wigner string operators $(\mathbb{1}-2n_j)$ commute with $n_{j'}$ and $(\mathbb{1}-n_{j'})$, for $j<i$ and $i<j'<i+r'$, and that

$$(\mathbb{1} - 2n_j)(\mathbb{1} - 2n_j) = \mathbb{1}, \tag{A5}$$

we can bring the Jordan-Wigner string $\Pi_{j<i}(\mathbb{1}-2n_j)$ associated with the annihilation operator $c_{i+r'}$ through the intervening spinless-fermion operators to obtain

$$\Big\langle c_i^\dagger (\mathbb{1} - n_{i+1}) \cdots n_{i+l_1} \cdots n_{i+l_p} \cdots (\mathbb{1} - n_{i+r'-1}) \times \prod_{i\le j<i+r'} (\mathbb{1} - 2n_j) c_{i+r'} \Big\rangle. \tag{A6}$$

Then, using the fact that

$$c_i^\dagger (\mathbb{1} - 2n_i) = c_i^\dagger,$$

$$n_j (\mathbb{1} - 2n_j) = -n_j,$$

$$(\mathbb{1} - n_j)(\mathbb{1} - 2n_j) = (\mathbb{1} - n_j), \tag{A7}$$

we can finally write the hardcore-boson expectation

$$\Big\langle b_i^\dagger \prod_{\text{empty}} (\mathbb{1} - n_j) \prod_{\text{filled}} n_j b_{i+r'} \Big\rangle = (-1)^p \Big\langle c_i^\dagger \prod_{\text{empty}} (\mathbb{1} - n_j) \prod_{\text{filled}} n_j c_{i+r'} \Big\rangle \tag{A8}$$

as a spinless-fermion expectation, where p is the number of occupied sites between i and $i+r'$. The suffixes "empty" or "filled" in the products in Eq. (A8) refer to the sites between i and $i+r$ which are empty or filled, respectively.

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[24] As we expect from having two flavors of bound pairs, the many-bound-pair ground state is twofold degenerate for ladders of even length L subject to periodic boundary conditions. For ladders of odd length L subject to periodic boundary conditions, the flavor of a bound pair changes as it goes around the boundary of the ladder, and so the conserved quantum numbers are not the even and odd flavors but are instead the symmetric and antisymmetric combinations of the two flavors. This mixing between even and odd flavors lifts the ground-state degeneracy giving a nondegenerate many-bound-pair ground state whose quantum number is the antisymmetric combination of flavors. In this paper we consider only ladders of even length L because we want to work with ground states containing bound pairs with a definite flavor.
[25] More quantitatively, every sector maps to an ordinary fermion chain such that each change in flavor (between successive pairs) diminishes the effective length L' by 1, thereby increasing the particle density (and hence the energy of that sector's ground state).
[26] The symmetry breaking has consequences for exact diagonalizations (Ref. 5). Since we always have the same number of spinless fermions on the two legs in this paired limit, we expect reflection about the ladder axis to be an exact symmetry of the ground states as well, as soon as $|t/t'|>0$ which permits a tiny tunnel amplitude between the even and odd sectors in finite ladders. The symmetrized ground states are $\frac{1}{\sqrt{2}}(|\Psi_+\rangle \pm |\Psi_-\rangle)$.
[27] The CDW-π correlations $\langle B_j^\dagger B_j B_{j+r}^\dagger B_{j+r}\rangle$ cannot be written as simple linear combinations of eight-point functions because a term like $\langle c_{2,j}^\dagger c_{1,j+1}^\dagger c_{1,j+1} c_{2,j} c_{2,j+r+1}^\dagger c_{1,j+r}^\dagger c_{1,j+r} c_{2,j+r+1}\rangle$ will pick up contributions from configurations that $\langle B_j^\dagger B_j B_{j+r}^\dagger B_{j+r}\rangle$ will not. This tells us that $\langle B_j^\dagger B_j B_{j+r}^\dagger B_{j+r}\rangle$ is some messy linear combination of 8-point, 12-point, 16-point, $\ldots$,$4n$-point functions.
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